**Savings-CAPM: A Possible Solution to the Consumption-CAPM Equity Premium Puzzle (EPP)**

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**Abstract**

MEHRA and PRESCOTT (1985) raised an issue that has still not yet been resolved in a satisfactory manner: the risk premium on US shares is (much) higher than could be explained by the neoclassic financial economics paradigm. Since then, this unresolved problem has become known as the Equity Premium Puzzle (EPP). This problem has triggered a series of papers, dissertations and theses that have attempted to adjust the expected intertemporal utility models to economic and financial data, especially related to the US market. However, these models, which are also called C-CAPM (Consumption-CAPM), have not been able to explain the aggregate behavior of consumers and the financial markets. This study presents a new intertemporal equilibrium model (S-CAPM) in an attempt to resolve the EPP, using the marginal savings utility instead of the marginal consumption utility, as they should be equal at each moment in time. Thus, this solution consists of a minor rearrangement of the models and the inclusion of macroeconomic information that has not been considered until now, such as the savings level and the per capita GDP. The mean risk aversion level obtained from these data (1929 and 2004) was below 10. Calculated through the approach adopted by Hansen and Jagannathan (1991), this risk aversion level was greater than or equal to 1.8.

**Introduction**

According to Campbell and Cochrane (2000), the development of the C-CAPM (Consumption-based Capital Asset Pricing Model) theory ranks among the main advances in financial economics over the past few decades. Classic papers by Lucas (1978), Breeden (1979) and then Grossman and Shiller (1981), among others, deduced that simple relations would have the power to explain complex intertemporal relations between the maximization of the expected marginal consumption utility and the expected rate of return on the financial assets. Thus, the C-CAPM presented disappointing results when compared with actual data.

During the 1980s, several studies indicated problems with the C-CAPM for explaining the development of the return on financial assets through the behavior of per capita consumption, such as Hansen and Singleton (1983) and Mehra and Prescott (1985). According to Mankiw and Shapiro (1986) and Breeden, Gibbons and Litzenberger (1989), the C-CAPM also did not prove more efficient than the traditional CAPM for forecasting the rates of return on financial assets. Consequently, the C-CAPM does not appear today in Finance handbooks, and it is not surprising that finance professionals no longer use it when taking investment decisions.

Based on the C-CAPM, Mehra and Prescott (1985) managed to explain the share risk premium if the risk aversion levels among investors were far higher than indicated by empirical and theoretical studies. Due to this inconsistency, this paper was entitled: ‘Equity Premium: A Puzzle’. The literature ended up by “adopting” this phrase by a minor modification: Equity Premium Puzzle (EPP) to describe this inconsistency.

Hansen and Jagannathan (1991) argued that the parametric approach of the utility functions, although offering interesting insights from the theoretical standpoint, might well be
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curtailing the explanatory capacity of the C-CAPM model in the empirical field. As a result, instead of attempting to explain the rates of return series for financial assets through a parametrized utility function, these authors did the opposite. Based on the rates of return series for financial assets, they deduced the general characteristics of these utility functions, or rather, the stochastic discount factor. However, no consumption utility function known at that time proved able to meet the requirements imposed by this methodology. This study ratified the existence of the Equity Premium Puzzle from the econometric standpoint.

Many academic studies were conducted in an attempt to solve this puzzle, outstanding among which are: Epstein and Zin (1989, 1991) and Campbell and Cochrane (1995). These studies found a possible solution for the EPP, basically modifying the consumption utility function formulation, in order to allow a higher risk aversion level with the capacity to explain the share risk premium. However, although the solutions presented so far have helped extend the knowledge of the academic community, according to Kocherlakota (1996), Mehra (2003), Mehra and Prescott (2003), Campbell et. al. (1997), among others, they have not yet been accepted as definitive.

This study attempts to (i) deduce the first order condition for maximizing the intertemporal utility agent, (ii) analyze the inconsistencies of this model in terms of actual data, which is the real puzzle, (iii) propose a modification to the model in order to obtain the solution to the EPP, and (iv) validate the logic of this model through the Mehra-Prescott methodology, and (v) the Hansen-Jagannathan methodology.

I. Deducing the first order condition for maximizing the intertemporal utility and the C-CAPM

In single-period maximization models, such as the traditional CAPM, it is assumed that the agent has already resolved the issue of how much of its initial income and wealth will be consumed and how much will be saved. Over a brief time span, agents will strive to maximize their expected wealth utility function through selecting investment alternatives and risk aversion levels for each of them.

However, it is unlikely that the agents will live only during a brief period of time. In addition, they will not consume all the wealth at the end of the first (brief) period, having nothing left for subsequent periods. In fact, investors must adapt their consumption level to their budget constraints, which are imposed by the income on their capital as well as from other sources (especially work). These distinct definitions have no impact on the first-order condition of the problem of maximizing the intertemporal utility.

Consequently, agents strive to maximize this utility throughout their lives, with the source of satisfaction (utility) being consumption. Wealth is endowed with utility only through constituting potential consumption. Thus, the goal of each agent is to maximize the following utility function:

$$\max \left[U\left(C_0\right) + J(W_0)\right] = \max \left[E \left[U\left(C_0; C_1; \ldots; C_{T-1}; W_T; t\right)\right]\right]$$  \hspace{1cm} (I.1)

In other words, by maximizing the sum of the current consumption utility and the current wealth utility, the agents are maximizing the expected consumption utility throughout their remaining lifetimes ($T$ periods). The wealth at the end of the period $T$, $W_T$, is the legacy left by each agent.

It is generally assumed that the consumption utility in a given period is independent of past or future consumption. Mathematically, this fact is represented by the following equation:
where \( \delta = 1/(1 + r) \), meaning that \( \delta \) would be a present value discount factor whose subjective discount rate would be \( r \). Thus, the time variable \((t)\) in the utility function serves to bring the future expected utilities to the present value through the \( \delta \) factor, with theoreticians defining \( 0 < \delta < 1 \). This parameter would be a subjective factor through which the agents decide to substitute R$ 1.00 of consumption today by R$ \((1/\delta)\) for consumption at some period in the future, or R$ \((1/\delta^t)\) in \( t \) periods. This \( \delta \) parameter is known as the ‘impatience’ level of the agents, also as the intertemporal substitution factor (ISF). It is generally assumed that this factor remains constant over time.

Still analyzing Equation (I.2) it is assumed that assumed that \( C_r = W_T \), meaning that the final consumption would be the bequeathed legacy.

Now it is necessary to include the intertemporal budget constraint in the model and then deduce the first order condition (Euler Equation) for maximizing the intertemporal utility. In the model deductions, it is assumed explicitly or implicitly that consumption – in addition to determining the current utility level of the agent – also plays the role of a control variable for attaining the optimum wealth level over time. In the following deduction, it is assumed that the agent, in addition to having a financial income brought in through its wealth inventory, also has income from other sources \((L)\). However, in order for consumption to play an exclusive role as a control variable, this non-financial income is exogenous, meaning that it is not controllable by the agent.

The budget constraint imposed on the agent consists of the sustainability of the expansion of its wealth in order to reach the planned level at the end of its life (or the expected survival horizon from a specific time onwards. As it is not known exactly how much the final wealth \((W_T)\) will be, nor are the rates of return known at which the wealth will be remunerated over time \((R)\), the intertemporal budget constraint appears as the outcome of the expectations that the representative agent considers to be credible and probable. Consequently, this constraint may be represented by the following equation:

\[
E(W_T) = E\left(W_0 \prod_{t=0}^{T-1}[1 + R_{t+1}] + E \sum_{t=0}^{T-1}(L_t - C_t) \prod_{s=0}^{T-1}[1 + R_{s+1}]\right)
\]

Thus, the expected wealth during the period \( T \) \((W_T)\) is the outcome of the expectations for the rates of return during \((R)\) for the accumulated wealth and savings \((L_t - C_t)\) expected over time.

Consequently, combining the objective function (I.2) with the budget constraint (I.3), the following Lagrangean function is obtained (conditional optimization):

\[
\Lambda = \sum_{t=0}^{T} \delta^t E[U(C_t)] + \lambda E\left(W_T - \sum_{t=0}^{T-1}(L_t - C_t) \prod_{s=0}^{T-1}[1 + R_{s+1}]\right)
\]

It is apparent that the first order conditions are identical in this case, when there is no income from work:

\[
\frac{\partial \Lambda}{\partial C_0} = \delta^0 E[U'(C_0)] + \lambda E\left(\prod_{s=0}^{T-1}[1 + R_{s+1}]\right) = 0 \Rightarrow U'(C_0) = -E\left(\lambda \prod_{s=0}^{T-1}[1 + R_{s+1}]\right)
\]

\[
\frac{\partial \Lambda}{\partial C_1} = \delta^t E[U'(C_t)] + \lambda E\left(\prod_{s=0}^{T-1}[1 + R_{s+1}]\right) = 0 \Rightarrow \delta E[U'(C_t)] = -E\left(\lambda \prod_{s=0}^{T-1}[1 + R_{s+1}]\right)
\]

In \( t = 0 \), the \( E(.) \) operator was omitted, because it is assumed that there is no uncertainty regarding the current consumption utility level. However, the expectation operator is not waived when \( t > 0 \), as there is uncertainty regarding the level and the current consumption utility. In general terms, the first order condition may be represented as shown...
below:
\[
\frac{\partial \Lambda}{\partial C_t} = \beta' E[U'(C_t)] + E \left( \lambda \prod_{s=t}^{T-1} [1 + R_{s+1}] \right) = 0 \Rightarrow \delta' E[U'(C_t)] = -E \left( \lambda \prod_{s=t}^{T-1} [1 + R_{s+1}] \right) \quad (I.5c)
\]

Rewriting Equation (I.5a) gives:
\[
U'(C_0) = -E \left( [1 + R_1] \lambda \prod_{s=1}^{T-1} [1 + R_{s+1}] \right) \quad (I.6)
\]

Substituting (I.5b) in (I.6), and assuming that \( \beta \) is constant, gives:
\[
U'(C_0) = \delta E \left( [1 + R_1] U'(C_1) \right) \quad (I.7)
\]

Thus, Equation (I.7) represents the equilibrium condition between current consumption \((t=0)\) and that for the next period. The marginal utility of avoiding consumption (saving) R$ 1.00 today must be equal to the expected marginal utility, discounted at present value (by the \( \delta \) factor) of consuming R$ \([1 + R_1]\) in the subsequent period \((t=1)\). However, the rate of return \( k_1 \) remunerates all the capital not consumed today \((t=0)\) through to the next period \((t=1)\). Consequently, it is only in \(t=1\) that \( R_1 \) will be a known rate. As the \( R_1 \) rate of return will affect the consumable wealth in \(t=1\), it will affect the decision on how much to consume during the next period.

This equilibrium condition is analogous to that deduced by Lucas (1978). However, this new form of demonstrating the equilibrium condition between the current period \((t=0)\) and the next period \((t=1)\) may be easily extended to the equilibrium condition of the consumption substitution between any two periods:
\[
E[U'(C_{t+1})] = \delta^{t_2-t_1} E \left[ U'(C_{t+1}) \prod_{s=t+1}^{t_2} [1 + R_{s+1}] \right] \quad \text{where } T \geq t_2 > t_1 \quad (I.8)
\]

It is worthwhile noting that, in this deduction, the fact that there is a non-financial income or not \((L, \text{ which is not under the control of the agent})\) does not alter the first order condition \((I.7)\) and \((I.8)\). Consequently, this outcome is rated as robust by the literature. Subsequently, this “non-controllable income” assumption will be questioned.

The implications of this model are explored below, with the findings of Mehra and Prescott.

II. Implications of the C-CAPM and the EPP

As already mentioned, Lucas (1978) developed an intertemporal asset tracking model in a barter (non-monetary) economy where there is only one company that produces perishable goods and a stock market where the shares of this company are traded competitively among the agents. In this economy, the agents do not need to decide on the composition of their investment portfolios, but are concerned only with determining current and future consumption levels. It is usually assumed that the present value of the legacy marginal utility \((W_T)\) is zero or, similarly, that \( T \) is infinite. Thus, the objective function of the agents is the following:
\[
\text{Max } E \left[ \sum_{j=0}^{\infty} \delta^j U(C_{t+j}) \right] \quad (II.1)
\]

Mehra and Prescott (1985) worked on the basis of this simplified model developed by Lucas (1978) and contributions from Grossman and Shiller (1981), among others, in order to deduce the theoretical implications of the intertemporal model and contrast them with real market data. When conducting this study, Mehra and Prescott
encountered a discrepancy between market data and current financial economics theory, coining a name for this discrepancy: Equity Premium Puzzle.

As seen in the previous section, the equilibrium condition for Equation (II.1) is the following:\(^1\)

\[
U'(C_i) = \partial E_t \left[ (1 + R_{t+1}) U'(C_{t+1}) \right] \tag{II.2}
\]

However, there is a difference between (II.2) and (I.7). While (II.2) \( R_{t+1} \) is the rate of return of any asset (or portfolio) \( i \) during the period \( t+1 \), \( R_t \) in (I.7) was defined as the rate of return for the entire portfolio held by the representative agent. Nevertheless, it is quite clear that the deduction from the model presented in the previous section may be extended to encompass any asset.

Dividing sides (II.2) by \( U'(C_i) \) gives:

\[
1 = \partial E_t \left[ \frac{(1 + R_{t+1}) U'(C_{t+1})}{U'(C_i)} \right] \tag{II.3}
\]

Most studies assume that the agents have a power utility function, or a constant relative risk aversion (CRRA).

\[
U(C_i) = \frac{C_i^{1+\gamma} - 1}{1 - \gamma} \tag{II.4}
\]

Where:
\( \gamma \) is the relative risk aversion coefficient. This risk aversion parameter does not depend on the wealth level of the agent.

Originally, the CRRA utility function (II.4) was developed using wealth (W) instead of consumption (C). However, many authors such as BREEDEN (1979), support the idea that there are no theoretical stumbling-blocks preventing the use of one variable or the other. As a result, the marginal benefit (utility) of the final centavo (R$ 0.01) consumed is obtained through a derivative of (II.4) as a function of C:

\[
U'(C_i) = C_i^{-\gamma} \tag{II.5}
\]

Thus, (II.3) may be re-written in the following manner:

\[
1 = \partial E_t \left[ \left( 1 + R_{t+1} \right) \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \tag{II.6}
\]

Assuming that the returns on assets and the consumption growth rate have a lognormal distribution, it may be demonstrated that \( \ln E[X] = E[\ln X] + \frac{1}{2} Var[\ln X] \). If \( X \) is equal to \( \left( 1 + R_{t+1} \right) \left( \frac{C_{t+1}}{C_t} \right)^{1-\delta} \) this relation may be rewritten as follows:

\[
\ln E_t \left[ \left( 1 + R_{t+1} \right) \left( \frac{C_{t+1}}{C_t} \right)^{1-\delta} \right] = E_t \left[ \ln \left( 1 + R_{t+1} \right) \left( \frac{C_{t+1}}{C_t} \right)^{1-\delta} \right] + \frac{1}{2} Var_t \left[ \ln \left( 1 + R_{t+1} \right) \left( \frac{C_{t+1}}{C_t} \right)^{1-\delta} \right] \tag{II.7}
\]

Thus, applying the \( \ln \) (natural logarithm) on both sides of (II.6), gives:
\[ 0 = \ln \delta + \ln \left( t + R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{\gamma} \right) \]

\[ = \ln \delta + E_t \left[ \ln \left( t + R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{\gamma} \right) \right] + \frac{1}{2} \text{Var}_t \left[ \ln \left( t + R_{t+1} \left( \frac{C_{t+1}}{C_t} \right)^{\gamma} \right) \right] \]

Defining that \( \Delta c_{t+1} = c_{t+1} - c_t = \ln C_{t+1} - \ln C_t = \ln \left( \frac{C_{t+1}}{C_t} \right) \) and that \( r_{t+1} = \ln(1 + R_{t+1}) \), results in:

\[ 0 = \ln \delta + E_t \left[ r_{t+1} - \gamma (\Delta c_{t+1}) \right] + \frac{1}{2} \text{Var}_t \left[ r_{t+1} - \gamma (\Delta c_{t+1}) \right] \]

\[ = \ln \delta + E_t \left( r_{t+1} - \gamma E_t (\Delta c_{t+1}) \right) + \frac{1}{2} E_t \left[ r_{t+1} - \gamma (\Delta c_{t+1}) \right]^2 - \left( E_t \left[ r_{t+1} - \gamma (\Delta c_{t+1}) \right] \right)^2 \]

\[ = \ln \delta + E_t \left( r_{t+1} - \gamma E_t (\Delta c_{t+1}) \right) + \frac{1}{2} E_t \left\{ \left( r_{t+1} \right)^2 - \left( \gamma E_t (\Delta c_{t+1}) \right)^2 \right\} - \gamma \left( E_t \left[ r_{t+1} \right] \right)^2 \]

Furthermore, defining that:

\[ \sigma_i^2 \quad r_{t+1} \text{ variance (natural logarithm of more than one rate of return on asset } \]
\[ i), \text{ or mathematically: } \sigma_i^2 = E_t \left[ (r_{t+1})^2 \right] - \left( E_t \left[ r_{t+1} \right] \right)^2 \]

\[ \sigma_e^2 \quad \Delta c_{t+1} \text{ variance (natural logarithm for one more consumption growth rate), or mathematically: } \sigma_e^2 = E_t \left[ (\Delta c_{t+1})^2 \right] - \left( E_t \left[ \Delta c_{t+1} \right] \right)^2 \]

\[ \sigma_{ie} \quad \text{covariance between } r_{t+1} \text{ and } \Delta c_{t+1} \text{ or mathematically: } \sigma_{ie} = E_t \left[ (r_{t+1})(\Delta c_{t+1}) \right] - \left( E_t \left[ r_{t+1} \right] \right) \left( E_t \left[ \Delta c_{t+1} \right] \right) \]

and, rearranging the terms, gives the following:

\[ E_t \left( r_{t+1} \right) = \gamma E_t \left( \Delta c_{t+1} \right) - \ln \delta - \frac{1}{2} \left( \sigma_i^2 + \gamma^2 \sigma_e^2 - 2 \gamma \sigma_{ie} \right) \quad (II.8) \]

For a risk-free asset (\( j \)), the variance in the returns as well as their covariance with consumption growth would be equal to zero:

\[ r_{j,t+1} = \gamma E_t \left( \Delta c_{t+1} \right) - \ln \delta - \frac{1}{2} \left( \gamma^2 \sigma_e^2 \right) \quad (II.9) \]

Thus, the risk premium may be obtained by subtracting (II.9) from (II.8), giving:
\[ E_i \left( r_{i,t+1} - r_{f,t+1} \right) + \frac{\sigma_i^2}{2} = \gamma \sigma_c \]  

(II.10)

As \( \sigma_i^2 \) is also equal to the \( \left( r_{i,t+1} - r_{f,t+1} \right) \) variance, as the risk-free asset variance is zero, and following the same principle used in (II.7), (II.10) may be re-written as follows:

\[ \ln E_i \left[ \frac{1 + R_{i,t+1}}{1 + R_{f,t+1}} \right] = \gamma \sigma_c \]  

(II.11)

Thus, Equation (II.11) shows that the risk premium of an asset \( i \) is a function of the risk aversion level times the covariance between the returns on asset \( i \) and consumption growth. However, as everything produced in this economy is consumed to the equilibrium point, and all dividend income paid to the shareholders comes from production profits, MEHRA (2003) imposes the equilibrium condition, where the consumption growth rate must be equal to the wealth growth rate provided by the shares \( \left( 1 + R_{i,t+1} \right) \), and thus: \( \sigma_i^2 = \sigma_{ic} = \sigma_c^2 \). According to MEHRA (2003), any of the variances or a covariance may be used, with this choice guided by the variance in the consumption growth rate. Thus, (II.12) is re-written as follows:

\[ \ln E_i \left[ \frac{1 + R_{i,t+1}}{1 + R_{f,t+1}} \right] = \alpha \sigma_c^2 \]  

(II.12)

According to Mehra and Prescott (1985), the \( \gamma \) parameter, which measures the wish to accept risk, is an important indicator in many economic fields. Its authors report countless studies \(^2\) that would provide \textit{a priori} justification, in which \( \gamma \) varies between zero and 2. On this basis, these authors established that any value above 10 might not be accepted without new empirical and theoretical evidence. The problem is that with the data available at that time (and still today), the \( \alpha \) needed to explain the share risk premium is very high. Inserting the value contained in this paper (and reproduced in Table II.1) into Equation (II.13) gives:

\[ \ln E_i \left[ \frac{1 + R_{i,t+1}}{1 + R_{f,t+1}} \right] = \gamma \sigma_c^2 \Rightarrow \ln \left[ \frac{1.0698}{1.0080} \right] = \gamma (0.001274) \]

\[ \gamma = 46.69 \]

Table II.1 extends the calculations for several sub-periods, using the data presented by MEHRA and PRESCOTT (1985, Table 1). It may be verified that \( \alpha \) was less than 10 in only two periods during the first decade analyzed, and during the decade subsequent to the crisis, the aversion level may be calculated in compliance with Equation (II.12).
<table>
<thead>
<tr>
<th>Period</th>
<th>Actual Consumption Growth per capita (%)</th>
<th>Rate of return on relatively risk-free asset (%)</th>
<th>S&amp;P 500 Rate (%)</th>
<th>E(γ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1889 - 1978</td>
<td>1.83% 3.57% 0.80% 5.67% 6.98% 16.54%</td>
<td>46.69</td>
<td>(III.1)</td>
<td></td>
</tr>
<tr>
<td>1889 - 1898</td>
<td>2.30% 4.90% 5.80% 3.23% 7.58% 10.02%</td>
<td>6.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1899 - 1908</td>
<td>2.55% 5.31% 2.62% 2.59% 7.71% 17.21%</td>
<td>17.17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1909 - 1918</td>
<td>44.00% 3.07% -1.63% 9.02% -0.14% 12.81%</td>
<td>15.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1919 - 1928</td>
<td>3.00% 3.97% 4.30% 6.61% 18.94% 16.18%</td>
<td>83.34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1929 - 1938</td>
<td>-25.00% 5.28% 2.39% 6.50% 2.65% 27.90%</td>
<td>0.91</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1939 - 1948</td>
<td>2.19% 2.52% -5.82% 4.05% 3.07% 14.67%</td>
<td>142.04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1949 - 1958</td>
<td>1.48% 1.00% -0.81% 1.89% 17.49% 13.08%</td>
<td>1693.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1959 - 1968</td>
<td>2.37% 1.00% 1.07% 0.64% 5.58% 10.59%</td>
<td>436.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1969 - 1978</td>
<td>2.41% 1.40% -0.72% 2.06% 0.03% 13.11%</td>
<td>38.40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: MEHRA and PRESCOTT (1985) and the author [column E(α)]

### III. Critical review of the C-CAPM assumptions and the S-CAPM deduction

Studies of intertemporal models such as the C-CAPM have invariably adopted the same assumptions presented by Lucas (1978). However, Lucas developed a model of a closed non-monetary economy with no government, in which a single type of perishable product was produced, meaning that it could not be stored. Thus, the identity of the national accounts of this economy would be as follows:

\[ Y_t = C_t \quad (\equiv W_t) \quad (III.1) \]

In other words, everything that is produced \( Y \) during a specific period must necessarily be consumed \( C \). As all the goods are perishable, the wealth at the start of the period (endowment) was fully consumed by the end of this period, or simply rotted. With this barter economy, the economic nature of variable wealth \( W \) was altered, which is a variable inventory in real life, and becomes a flow variable.

This formulation is attractive to economists, because it is difficult to estimate the per capita wealth inventory in an economy. And in this artificial economy, it may be assumed that the agents are always in balance, with the marginal consumption utility is equal to the marginal wealth utility:

\[ U'(W_t) = U'(C_t) \quad (III.2) \]

Through this, all types of utility functions may be applied to wealth for consumption. Assuming a function of the CRAA-type function, gives:

\[ W_t^{-\gamma} = C_t^{-\gamma} \quad (III.3) \]

However, according to macro-economic handbooks such as that written by Dornbusch et. al. (1998), in a modern open economy, there is a government and a broad range of goods being produced, consumed and stored (invested). Thus, the wealth generated during a period may be extended as follows:

\[ Y_t = C_t + I_t + G_t + (X_t - M_t) \quad (III.4) \]

All the wealth produced in the period \( Y \), Gross Domestic Product) may be consumed by individuals \( C \) or by the government \( G \) and by non-residents \( X \), exports) with a part “stored” in the form of machines, facilities or even goods not yet consumed \( I \), investment). On the other hand, parts of the expenditures of residents and the government
(C+I+G) are allocated to goods and services produced outside this economy. In order to avoid double accounting, imports (M) appear in the equation with a negative sign. In a modern economy, in a modern open economy, is it possible to explain the rates of return on assets based only on the consumption of a “representative agent”?

In addition to the expenditure standpoint (III.4), income may also be analyzed from the allocation standpoint:

\[ Y_t = C_t + S_t + T_t - TR_t \]  \hspace{1cm} (III.5)

All the income (Y) generated by the economy is either consumed (C) or is saved (S), or is collected by the government in the form of taxes and levies (T). However, part of what is collected by the government returns to the private sector in the form of transfers (TR), which are pensions for retirees, social welfare programs and subsidies for some producers. In order to avoid double counting, transfers are shown in Equation (IV.5) with a negative sign.

From the private sector viewpoint, the available income (YD) for consumption or saving is only:

\[ YD_t = Y_t - T_t + TR_t \]  \hspace{1cm} (III.6)

Thus, the allocation of the available income demarcates only consumption and savings:

\[ YD_t = C_t + S_t \]  \hspace{1cm} (III.7)

Equaling out equations (III.4) and (III.5), it is possible to perceive the role of information contained by gross private savings:

\[ S_t = I_t + (G_t + TR_t - T_t) + (X_t - M_t) \]  \hspace{1cm} (III.8)

The gross savings of the private sector (S) in any period are always equal to the gross investments (I) plus the nominal government deficit (G+TR-T) plus the current transaction surplus (X-M). Thus, it may be noted that the gross savings by the private sector (S), in addition to reflecting the relation between gross income and Private Consumption (YD-C) is also subject to the impact of fluctuations in gross investments (I), the nominal government deficit (G+TR-T) and also the current transaction surplus (X-M). These inter-relations are not addressed in the C-CAPM models developed to date.

Assuming that the agents achieve satisfaction (utility) through current and future consumption, with the latter represented by current savings, it is possible to make some deductions about the relation between the marginal consumption utility and the marginal savings utility based on Equation (III.7). To do so, it is sufficient to optimize the utility as a consumption and current savings, subject to the budget constraint represented by Equation (III.7). This gives:

Maximize \( U(C_t; S_t) \) subject to \( YD_t = C_t + S_t \).

Applying the Lagrange multiplicative methodology, gives:

\[ L = U(C_t; S_t) + \lambda (YD_t - C_t - S_t) \]  \hspace{1cm} (III.9)

Whose first order conditions are:

\[ L_C = U_C(C_t; S_t) - \lambda = 0 \Rightarrow U_C = \lambda \]  \hspace{1cm} (III.10a)

\[ L_S = U_S(C_t; S_t) - \lambda = 0 \Rightarrow U_S = \lambda \]  \hspace{1cm} (III.10b)

\[ L_\lambda = YD_t - C_t - S_t = 0 \]  \hspace{1cm} (III.10c)

Similarly, equaling (III.10a) and (III.10b) gives:

\[ U_C = U_S \]  \hspace{1cm} (III.11)

In other words, at each instant of time, the agent striving to maximize its wealth utility will consume up to the level where the marginal consumption utility is equal to that of savings. Should it consume more (or less) than this level, it will not be at its optimum point,
even complying with the budget constraint.

Graph III.1. portrays the situation of an agent with current wealth ($W_0$) of $200 and an available income (YD) of $100. Among countless indifference curves, the U(0) curve is that in which the agent utility is maximized, taking budget constraints into account.

**Graph III.1. Trade-off between Consumption and Current Savings**

It is worthwhile noting that, from this standpoint, current wealth does not represent any constraint. The agents in a closed economy (and with no government) may not have negative gross savings in aggregate terms, unless they can consume the accumulated wealth (W). However, consumption during a specific period would be limited to $C_r = YD_r - S_r \leq YD_r + W_r$. E, and as there can be no negative consumption, savings would be limited to the interval $YD_r \geq S_r \geq -W_r$.

On the other hand, in an open economy, consumption during a given period would not be limited, as consumption could be financed through a deficit in current transactions, $(X_t - M_t) < 0$. Consequently, savings would also have no lower limit.

Based on this conclusion, the marginal savings utility is always equal to the marginal consumption utility, it may be accepted that the current savings utility may be an indirect measurement of the instant satisfaction of the agents with some advantages. The most important of them lies in the fact that not all outlays on consumption are fully rational, or rather discretionary. There are several items required to meet their basic needs that the agents cannot even consider avoiding. Thus, consumption is not a “pure” control variable, as it is not possible to distinguish between the discretionary and non-discretionary portions. A line of research appeared [Constantinides (1990), Sundaresan (1989), Abel (1990, 1996) and Campbell and Cochrane (1999)] to explore this approach, which is the consumption habits formation hypothesis.

This study attempts to surmount this issue of discretionary consumption through the use of savings, as this part of the current income allocation may be viewed as fully discretionary. Moreover, current savings are parts of the income that are exposed to financial
risks, while parts of the current income that are consumed are not subject to these risks. Thus, from this standpoint, it makes no sense to apply the CRRA (constant relative risk aversion) utility model to consumption, but it is quite logical to apply it to savings.

Savings are frequently viewed as the unconsumed portion of income, and assuming that the income is not controllable, consumption is the only control variable. But what our reflections and research may well be underestimating is the savings capacity of societies, not through reducing consumption, but rather through increasing income, by trying harder to produce goods and services through more hours worked, for example. In this case, savings become a control variable that encompasses both consumption and income.

Moreover, it is savings rather than consumption that drive demands for financial assets. The supply of financial assets is determined by investment (I), budget deficit (G+TR−T) or Private Consumption (C). However, the demand for financial assets is driven only by savings (S). From this standpoint, the parametrization of the marginal utility from the savings viewpoint seems to offer better potential for explaining the behavior of returns (and prices) of financial assets.

The outstanding balance for current transactions (X−M) may also function on the financial assets demand side, in case of a deficit (X<M), as well as on the supply side for these assets, in case of a surplus (X>M). H would increase only when there is a surplus, as wealth tends to shrink when there is a deficit, because household savings are unable to finance excess demand. However, if the global economy is analyzed in an aggregate manner, these imbalances in the current transactions account vanish.

Thus, this study proposes an alteration in the objective function, which must be viewed as an indirect way of obtaining the same original objective function. Thus, instead of Equation (II.1), which strives to maximize the consumption utility over time, the objective function becomes the following:

$$\text{Max} E\left[ \sum_{t=0}^{\infty} \delta^t U\left(S_{t+j}\right) \right]$$  \hspace{1cm} (III.12)

This new function may be construed as the best way of maximizing the expected utility in not consuming the income generated in each period. It is thus this unconsumed portion of the income that is subject to financial risks and also contains an intertemporal component. Consumption is always immediate.

The intertemporal budget constraint is the same as that which was applied in Section I, although explicitly containing savings:

$$E(W_T) = E\left( W_0 \prod_{t=0}^{T-1} [1 + R_{t+1}] \right) + E\left( \sum_{t=0}^{T-1} (S_t) \prod_{t=0}^{T-1} [1 + R_{t+1}] \right)$$  \hspace{1cm} (III.13)

Applying the Lagrangean Function gives:

$$\Lambda = \sum_{t=0}^{T} \delta^t E[U(S_t)] + \lambda \left\{ E(W_T) - E\left( W_0 \prod_{t=0}^{T-1} [1 + R_{t+1}] \right) - E\left( \sum_{t=0}^{T-1} (S_t) \prod_{t=0}^{T-1} [1 + R_{t+1}] \right) \right\}$$  \hspace{1cm} (III.14)

It is apparent that the first order conditions, in this case, are analogous with those found in Section I:
\[
\frac{\partial \Lambda}{\partial S_0} = \delta^0 E[U'(S_0)] - \lambda E\left(\prod_{s=0}^{T-1}[1 + R_{s+1}]\right) = 0 \quad \Rightarrow \quad U'(S_0) = E\left(\prod_{s=0}^{T-1}[1 + R_{s+1}]\right) \tag{III.15a}
\]

\[
\frac{\partial \Lambda}{\partial S_t} = \delta^t E[U'(S_t)] - \lambda E\left(\prod_{s=1}^{T-1}[1 + R_{s+1}]\right) = 0 \Rightarrow \delta E[U'(S_t)] = -E\left(\frac{\lambda}{\prod_{s=1}^{T-1}[1 + R_{s+1}]\right) \tag{III.15b}
\]

In \( t = 0 \), the \( E(.) \) operator was omitted because it is assumed that there is no uncertainty regarding the marginal current consumption utility and level, and consequently that of savings. However, the expectation operator is not waived when \( t > 0 \). as there is uncertainty regarding savings and consumption level utilities. In general terms, the first order condition may be represented as follows:

\[
\frac{\partial \Lambda}{\partial C_t} = \delta^t E[U'(C_t)] + E\left(\frac{\lambda}{\prod_{s=t}^{T-1}[1 + R_{s+1}]\right) = 0 \Rightarrow \delta E[U'(C_t)] = -E\left(\frac{\lambda}{\prod_{s=t}^{T-1}[1 + R_{s+1}]\right) \tag{III.15c}
\]

Rewriting Equation (III.15a) gives:

\[
U'(S_0) = -E\left(\frac{1 + R_1}{\prod_{s=1}^{T-1}[1 + R_{s+1}]\right) \tag{III.16}
\]

Substituting (III.15b) in (III.16), and assuming that \( \beta \) is constant, gives:

\[
U'(S_0) = \delta E\left[1 + R_1, U'(S_1)\right] \tag{III.17}
\]

Comparing Equation (III.17) with (I.7), it is apparent that the marginal consumption utility was replaced by the marginal savings utility. In the next section, the implications of the S-CAPM are assessed, in the approach adopted by Mehra and Prescott.

### IV. Implications of the S-CAPM in the Mehra-Prescott Approach

Adapting Equation (III.17) to any asset (or asset portfolio) \( i \), gives:

\[
U'(S_t) = \delta E\left[1 + R_{t,x+1}, U'(S_{t+1})\right] \tag{IV.1}
\]

This equation indicates that the marginal savings utility for R$ 1.00 during period \( t \), \( U'(S_t) \), is equal to the present value of the expected marginal consumption utility (which is equal to that of the savings) \( 1 + R_{t,x+1} \) in R$ during period \( t+1 \). The present value is obtained by applying an intertemporal discount factor \( \delta \) and \( R_{t,x+1} \) is the rate of return on a financial asset (fixed income paper or share) between periods \( t \) and \( t+1 \).

Dividing both sides of (IV.1) by \( U'(S_t) \) gives:

\[
1 = \delta E\left[1 + R_{t,x+1}, \frac{U'(S_{t+1})}{U'(S_t)}\right] \tag{IV.2}
\]

The sequence in this section is a consequence of this alteration to the classic model developed by Mehra and Prescott (1985) and the discussion on the parametrization of the savings utility function. As discussed in the previous section, it is assumed that the savings utility is of the Constant Relative Risk Aversion (CRRA) type:
\[ U(S_t) = \frac{(S_t)^{\gamma}}{1-\gamma} \]  

(IV.3)

Consequently, the marginal savings utility is as follows:

\[ U'(S_t) = (S_t)^{\gamma} \]  

(IV.4)

Thus, rewriting Equation (IV.2) based on (IV.4) gives:

\[ 1 = \delta E_t \left[ \left( 1 + R_{i,t+1} \right) \left( \frac{S_{i,t+1}}{S_t} \right)^{\gamma} \right] \]  

(IV.5)

From this point onwards, the model derivation follows very closely the model analyzed in Section I. Assuming that the rate of return and the private savings growth rate follow a joint lognormal and homoscedastic distribution, facilitates the analysis of Equation (IV.5) through applying the same logarithmic transformation that was used in Equation (II.6) and that gave rise to (II.8). Thus, applying the logarithmic transformation to both sides of Equation (IV.5), and recalling that the lower-case variables represent the natural logarithm (ln) gives:

\[ E_t \left[ r_{i,t+1} \right] = -\ln \delta + \gamma E_t \left[ \Delta s_{i,t+1} \right] - \frac{1}{2} \left[ \sigma_i^2 + \gamma^2 \sigma_s^2 - 2\gamma \sigma_{is} \right] \]  

(IV.6)

Where \( \sigma_i^2 \) means the variance \( \ln(1+R_{i,t+1}) = r_{i,t+1} \), \( \sigma_s^2 \) variance is the \( \ln \left( \frac{S_{i,t+1}}{S_t} \right) = \Delta s_{i,t+1} \) variance and \( \sigma_{is} \) is the covariance of these logarithms.

In the case of a risk-free rate where the variance and covariance (with any other variable) are equal to zero, the equation analogous to (IV.6) would be as follows:

\[ r_{f,t+1} = -\ln \delta + \gamma E_t \left[ \Delta s_{t+1} \right] - \frac{1}{2} \left[ \gamma^2 \sigma_i^2 \right] \]  

(IV.7)

Thus, if (IV.6) is subtracted from (IV.7), an estimated risk premium is obtained for any asset (assuming that the return and savings growth rates follow lognormal distributions):

\[ E_t \left[ r_{i,t+1} - r_{f,t+1} \right] + \frac{\sigma_i^2}{2} = \gamma \sigma_{is} \]  

(IV.8)

The variance term on the left side of Equation (IV.8) is due to working with logarithm expectations of \( (1+R) \). Recalling that \( \left\{ \ln E[X] = E[\ln X] + \frac{1}{2} \text{Var}[\ln X] \right\} \) and equaling \( X \) to \( \left[ \frac{1+R_{i,t+1}}{1+R_{f,t+1}} \right] \), Equation (IV.8) may be rewritten as follows:

\[ \ln \left[ E_t \left[ \frac{1+R_{i,t+1}}{1+R_{f,t+1}} \right] \right] = \gamma \sigma_{is} \]  

(IV.9)

Thus, according to Equation (IV.9), the risk premium for any asset is obtained through the product between the risk aversion level and the covariance between the rates of return of this asset and the aggregate savings.

Equations (IV.8) and (IV.9) will be used to estimate the relative risk aversion (\( \gamma \)) by the S-CAPM. The next Section explains how the data were obtained and handled, in order to reach the estimates presented in Section IV.4.
IV.1. Sources of data used in this study

The data used in this study are the following:
Annual macro-economic data from the USA (1929-2004):
(i) Private Savings, (ii) Private Consumption, and (iii) Available Income
Consumer Price Index (CPI)
Population
Data on rates of return on the US market: (i) S&P 500 and (ii) Yield on US Treasury Papers – 1 year.

The macro-economic data were taken from the Bureau of Economic Analysis (BEA, [http://www.bea.gov/](http://www.bea.gov)). The data used in the study were obtained from the following National Income and Product Accounts (NIPA) Table:

<table>
<thead>
<tr>
<th>Series</th>
<th>NIPA Table</th>
<th>Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross Private Savings (S)</td>
<td>5.1</td>
<td>+ 3 (Net Private Savings)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+14 (Private Consumption of Capital Goods)</td>
</tr>
<tr>
<td>Private Consumption (C)</td>
<td>1.1.5. Annual</td>
<td>+ 2 (Personal Consumption Expenditure)</td>
</tr>
<tr>
<td>Available Gross Income (YD)</td>
<td></td>
<td>Sum of the above lines</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(YD=C+S)</td>
</tr>
</tbody>
</table>

All these series are nominal, meaning they are expressed in current currency for each year between 1929 and 2004. In this paper, it was decided to work with all the nominal macro-economic and financial data and then convert them to a single Consumer Price Index (CPI) rather than by the implicit US GDP deflator (or that of some other sub-account). The Consumer Price Index – All Urban Areas was obtained from the website of the Bureau of Labor Statistics (BLS, [http://www.bls.gov/](http://www.bls.gov)).

The population used to calculate the per capita values was that estimated for July 1 each year by the US Census Bureau ([http://www.census.gov/](http://www.census.gov)).

Finally, data on the historic S&P Composite rates of return adjusted for dividends were obtained from the website of Professor Robert Shiller ([http://www.econ.yale.edu/~shiller/data.htm](http://www.econ.yale.edu/~shiller/data.htm)), through which he provides data on the US economy since 1871, frequently updated and reviewed. The name of the MS-Excel file containing these data is: chapt26.xls. This file also provides the short-term (1 year) and long-term (10 years) interest rates. In order to estimate the risk-free rate for calculating the market risk premium, the short-term rate is used in this paper.

IV.2. Findings obtained in the Mehra-Prescott Structure

The following calculations were made from the collected data as required to transform the macro-economic variables series into per capita terms, expressed in US dollars and constant purchasing power. Moreover, the real rates of return were obtained, eliminating the influence of inflation. Based on the values obtained, the moments of each variable were calculated, as well as the covariances and correlations between each pair of variables, with these findings summarized in Table IV.1.
Table IV.1. Growth Moments for Consumption, Savings, Income and Return on Assets

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔC/C</td>
<td>1.92%</td>
<td>3.79%</td>
<td>1.99%</td>
<td>7.88%</td>
<td>1.36%</td>
</tr>
<tr>
<td>ΔS/S</td>
<td>3.54%</td>
<td>20.46%</td>
<td>4.84%</td>
<td>18.91%</td>
<td>4.05%</td>
</tr>
<tr>
<td>ΔYD/YD</td>
<td>12.55%</td>
<td>418.66%</td>
<td>23.44%</td>
<td>357.73%</td>
<td>16.36%</td>
</tr>
<tr>
<td>RM</td>
<td>12.55%</td>
<td>29.37%</td>
<td>12.75%</td>
<td>42.57%</td>
<td>-4.16%</td>
</tr>
<tr>
<td>RF</td>
<td>29.37%</td>
<td>418.66%</td>
<td>84.13%</td>
<td>86.70%</td>
<td>111.80%</td>
</tr>
<tr>
<td>[(1+RM)/(1+RF)]-1</td>
<td>45.18%</td>
<td>111.80%</td>
<td>43.60%</td>
<td>356.12%</td>
<td>19.50%</td>
</tr>
<tr>
<td>Correlation</td>
<td>1</td>
<td>0.47</td>
<td>0.76</td>
<td>0.64</td>
<td>-0.27</td>
</tr>
<tr>
<td>ΔC/C</td>
<td>0.47</td>
<td>1</td>
<td>0.89</td>
<td>0.28</td>
<td>-0.40</td>
</tr>
<tr>
<td>ΔS/S</td>
<td>0.76</td>
<td>0.89</td>
<td>1</td>
<td>0.45</td>
<td>-0.38</td>
</tr>
<tr>
<td>ΔYD/YD</td>
<td>0.64</td>
<td>0.28</td>
<td>0.45</td>
<td>1</td>
<td>-0.05</td>
</tr>
<tr>
<td>RM</td>
<td>-0.27</td>
<td>-0.40</td>
<td>-0.38</td>
<td>-0.05</td>
<td>1</td>
</tr>
<tr>
<td>RF</td>
<td>0.68</td>
<td>0.36</td>
<td>0.51</td>
<td>0.98</td>
<td>-0.26</td>
</tr>
<tr>
<td>[(1+RM)/(1+RF)]-1</td>
<td>0.68</td>
<td>0.36</td>
<td>0.51</td>
<td>0.98</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Legend: $5\% = \frac{5}{100} = 0.05$ while $5\%^2 = \frac{5^2}{100^2} = \frac{5}{10000} = 0.0005$

It is worthwhile noting that the consumption growth rate (ΔC/C) shown in Table IV.1 has a correlation with the share risk premium ([1+RM]/[1+RF]-1) of 0.68, which is higher than the correlation between the premium and the savings growth rate (ΔS/S), which is 0.36. However, the covariance between ([1+RM]/[1+RF]-1) and ΔC/C is $45.18\%^2$, which is less than half the covariance between the premium and ΔS/S, which is $111.80\%^2$.

This fact indicates that the path followed by the S-CAPM is promising, at least as an attempt to solve the EPP, as the low historical volatility of consumption is frequently mentioned as the “quantitative clause” of the existence of the Equity Premium Puzzle [Mehra (2003)].

It is worthwhile recalling that, in order to deduce the S-CAPM in Section IV.1, it was assumed that the savings growth and return rates follow a joint lognormal distribution. However, it is necessary to obtain the logarithm moments for the savings growth and return rates (increased by 1). These findings are presented in Table IV.2.
Table IV.2. Logarithm Moments for Growth in Consumption, Savings, Income and Return on Assets

<table>
<thead>
<tr>
<th></th>
<th>Natural Logarithm 1 + the following rates</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Δc</td>
<td>Δs</td>
<td>Δyd</td>
<td>rM</td>
<td>rf</td>
<td>rM - rf</td>
</tr>
<tr>
<td>Mean</td>
<td>1.84%</td>
<td>1.90%</td>
<td>1.85%</td>
<td>5.99%</td>
<td>1.27%</td>
<td>4.72%</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>3.54%</td>
<td>19.49%</td>
<td>4.86%</td>
<td>18.30%</td>
<td>4.01%</td>
<td>18.93%</td>
</tr>
<tr>
<td>Variance</td>
<td>12.52%^2</td>
<td>379.77%^2</td>
<td>23.57%^2</td>
<td>335.06%^2</td>
<td>16.12%^2</td>
<td>358.19%^2</td>
</tr>
<tr>
<td>Covariance</td>
<td>Δc</td>
<td>12.52%^2</td>
<td>32.46%^2</td>
<td>13.08%^2</td>
<td>41.73%^2</td>
<td>-3.88%^2</td>
</tr>
<tr>
<td></td>
<td>Δs</td>
<td>32.46%^2</td>
<td>379.77%^2</td>
<td>84.49%^2</td>
<td>100.76%^2</td>
<td>-31.19%^2</td>
</tr>
<tr>
<td></td>
<td>Δyd</td>
<td>13.08%^2</td>
<td>84.49%^2</td>
<td>23.57%^2</td>
<td>39.71%^2</td>
<td>-7.47%^2</td>
</tr>
<tr>
<td></td>
<td>rM</td>
<td>41.73%^2</td>
<td>100.76%^2</td>
<td>39.71%^2</td>
<td>335.06%^2</td>
<td>-3.51%^2</td>
</tr>
<tr>
<td></td>
<td>rM - rf</td>
<td>45.61%^2</td>
<td>131.95%^2</td>
<td>47.17%^2</td>
<td>338.57%^2</td>
<td>-19.62%^2</td>
</tr>
<tr>
<td>Correlation</td>
<td>Δc</td>
<td>1</td>
<td>0.47</td>
<td>0.76</td>
<td>0.64</td>
<td>-0.27</td>
</tr>
<tr>
<td></td>
<td>Δs</td>
<td>0.47</td>
<td>1</td>
<td>0.89</td>
<td>0.28</td>
<td>-0.40</td>
</tr>
<tr>
<td></td>
<td>Δyd</td>
<td>0.76</td>
<td>0.89</td>
<td>1</td>
<td>0.45</td>
<td>-0.38</td>
</tr>
<tr>
<td></td>
<td>rM</td>
<td>0.64</td>
<td>0.28</td>
<td>0.45</td>
<td>1</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>rf</td>
<td>-0.27</td>
<td>-0.40</td>
<td>-0.38</td>
<td>-0.05</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>rM - rf</td>
<td>0.68</td>
<td>0.36</td>
<td>0.51</td>
<td>0.98</td>
<td>-0.26</td>
</tr>
</tbody>
</table>

Legend: $5\% = \frac{5}{100} = 0.05$ while $5\%^2 = \frac{5}{100^2} = \frac{5}{10,000} = 0.0005$

The covariance of the risk premium and the consumption growth rate were practically equal to those in Table IV.1, at 45.18%^2 for 45.61%^2. Meanwhile, the savings growth rate covariance with the share premium was even higher, rising to 131.95%^2, almost three times more than the consumption covariance.

As seen previously, the compatible relative risk aversion level ($\gamma$) for the historic share premium risk may be obtained through Equations (IV.8) and (IV.9), initially estimated at $\gamma$ through rearranging Equation (IV.8):

$$\gamma = \frac{1}{\sigma_{Ms}^2} \left[ E\left( r_{M,t+1} - r_{f,t+1} \right) + \frac{\sigma_M^2}{2} \right]$$  \hspace{1cm} (IV.10)

The problem with this calculation is that the “risk-free rate”, which should theoretically have a standard deviation of close to zero, presents a value that is almost four times higher than its mean. Thus, $\gamma$ will be estimated using the market covariance (S&P 500) with the savings growth logarithm ($\sigma_{Ms}$), together with the risk premium covariance with the
savings growth logarithm$\left(\sigma_{(M-f)}\right)$. Consequently, the market rate of return variance in Equation (IV.10) will be replaced by the value obtained for $\sigma_{M}^2$ as well as by the risk premium variance $\sigma_{(M-f)}^2$.

Table IV.3 shows the estimated value of each combination between the covariance value ($\sigma_{Ms}$) and the variance ($\sigma_{M}^2$) used in Equation (IV.10).

Table IV.3. Estimates for the Relative Risk Aversion Level ($\gamma$) through Equation (IV.10)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma_{M}^2$</th>
<th>$\sigma_{(M-f)}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{Ms}$</td>
<td>6.343</td>
<td>6.458</td>
</tr>
<tr>
<td>$\sigma_{(M-f)s}$</td>
<td>4.843</td>
<td>4.931</td>
</tr>
</tbody>
</table>

The underscored values in boldface are those that must be accepted, as they present coherence between the variance and covariance to be used in Equation (IV.10). Even so, the relative risk aversion level ($\gamma$) was not located outside the range established by Mehra and Prescott (1985) in any of the combinations, which when minimally acceptable, is between zero and 10.

However, it is still necessary to ascertain whether the subjective intertemporal substitution factor ($\delta$) estimated through Equation (IV.6) is positive and less than 1. Rearranging the terms of Equation (IV.6) gives:

$$\ln \delta = -\frac{\sigma_i^2}{2} \gamma^2 + [E_i(\Delta s_{t+1}) + \sigma_u] \gamma - E_i[r_{t+1}] - \frac{\sigma_i^2}{2}$$

$$\delta = e^{-\frac{\sigma_i^2}{2} \gamma^2 + [E_i(\Delta s_{t+1}) + \sigma_u] \gamma - E_i[r_{t+1}] - \frac{\sigma_i^2}{2}}$$

Through Table IV.2, all the variables are known, with the exception of $\delta$ and $\gamma$. Thus, Graph IV.1 outlines the $[\delta=f(\gamma)]$ functions, taking three asset portfolios into consideration: the S&P 500 portfolio (M, dark blue dotted line), the portfolio whose return pursues the S&P 500 risk premium (M-f, in red) and the “risk-free” assets portfolio (f, in pink). The risk premium and S&P 500 curves are very close, below the “risk-free” portfolio curve.

It is worthwhile noting that the higher the $\gamma$ the lower is $\delta$. This means that for a higher risk aversion level ($\gamma$), the lower the subjective intertemporal discount factor ($\delta$). Inserting the $\gamma$ values from Table IV.3 into Equation (IV.11) gives the $\delta$ values as shown in Graph IV.3.
It is worthwhile noting that throughout the entire possible relative risk aversion level (γ) spectrum, and in all portfolios analyzed, the subjective intertemporal substitution factor (δ) is not greater than 1. This virtually eliminates the possibility of having what is known as a Risk-Free Rate Puzzle in the S-CAPM Model. This “new” puzzle arises when δ>1, which is a paradox, as the agents would be subjectively deducting the future expected utilities at a negative rate, meaning the present value of the expected consumption utility R$ 1.00 in future will be greater than R$ 1.00! Graph IV.3 shows that, based on the sample used in this study, no problem was detected related to the Risk-Free Rate Puzzle.

Now the γ values will be obtained through Equation (IV.9). Rearranging this Equation gives:

\[
\gamma = \frac{1}{\sigma_{Ms}} \ln \left( \frac{1 + R_{M,t+1}}{1 + R_{f,t+1}} \right) \tag{IV.12}
\]

Consulting Table IV.1, it is apparent that the value estimated for \( E_t \left[ \frac{1 + R_{M,t+1}}{1 + R_{f,t+1}} \right] \) is 1.0602. Using the same covariance definitions as in Table IV.3 leads to Table IV.4:

<table>
<thead>
<tr>
<th>( \sigma_{Ms} )</th>
<th>( \sigma_{Ms} )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.187</td>
<td>4.725</td>
<td></td>
</tr>
</tbody>
</table>

The values estimated for γ remain within the established range of zero to 10.

Now the subjective intertemporal substitution factors will be estimated through
Equation (IV.7), which discards the variance and covariance terms involving the risk-free portfolio. Rearranging Equation (IV.7) gives:

$$\ln\delta = -\frac{\sigma^2}{2} \gamma^2 + E_t[\Delta s_{t+1}] \gamma - r_{f,t+1}$$

$$\delta = e^{-\frac{\sigma^2}{2} \gamma^2 + E_t[\Delta s_{t+1}] \gamma - r_{f,t+1}}$$

(IV.13)

Graph IV.2 shows the possible $\delta$ values as a function of $\gamma$ through Equation (IV.13). Moreover, the $\delta$ values are shown corresponding to the $\gamma$ values obtained from Table IV.4. Using the $\gamma$ obtained through the S&P 500 Portfolio ($\gamma = 6.187$), a $\delta$ value of 0.537 is obtained. If the $\gamma$ obtained from the portfolio is used, pursuing the risk premium ($\gamma = 4.725$), a $\delta = 0.707$ value is obtained.

**Graph IV.2. Relation between $\gamma$ and $\delta$ through Equation (IV.13)**

Similar to Graph IV.1, Graph IV.2 shows that $\delta$ is always less than 1. Thus, the Risk-Free Rate Puzzle does not even arise through Equation (IV.13) and the sample data.

In this Section, estimates were drawn up for the relative risk aversion level ($\gamma$) of the savings utility function, varying between 4.72 and 6.34. The values obtained thus fall within the $0 < \gamma < 10$ range established by Mehra and Prescott (1985). It is worthwhile stressing that the values estimated for the subjective intertemporal substitution factor ($\delta$) also fall within the “rational” range ($0 < \delta < 1$), more specifically between 0.457 and 0.707.

Despite the apparent success of the S-CAPM in solving the Equity Premium Puzzle, it must still be validated. This validation is presented in the next Section, based on the work of Hansen and Jagannathan (1991).

It is worthwhile recalling that the findings presented in this Section are dependent on the assumption that the rates of return for the assets and the savings growth rates follow a joint lognormal distribution. As the Hansen-Jagannathan methodology does not use any assumption for the utility function parametrization, nor its probability distribution, this may be considered as a good test for validating not only the values obtained in this Section but
above all the S-CAPM assumptions.

V. Implications of S-CAPM in the Hansen-Jagannathan Approach

This Section attempts to validate the S-CAPM assumptions within the methodological structure developed by Hansen and Jagannathan (1991). This validation will be conducted through comparing the S-CAPM implications with the lower volatility threshold of the stochastic discount factor \( (M^*_t) \), which was unknown to the authors.

The intertemporal stochastic discount factor (M) was defined by Hansen and Jagannathan (1991) as:

\[
M_t = \left( \frac{\delta U'(C_{t+1})}{U'(C_t)} \right)
\]

(V.1)

Within the S-CAPM context, this must be redefined as follows:

\[
M_t = \left( \frac{\delta U'(S_{t+1})}{U'(S_t)} \right)
\]

(V.2)

The first-order condition for maximizing the intertemporal utility of Lucas (1978) may be represented as shown below:

\[
t = E_t \left[ (t + R_{t+1}) M_{t+1} \right]
\]

(V.3)

Where:

- \( t \) is an N-sized vector containing only figures 1, \( t = [1 \ 1 \ \ldots \ 1]^T \)
- \( R_{t+1} \) is a vector containing the rates of return on N assets,
  \[
  R_{t+1} = \begin{bmatrix} R_{1,t+1} & R_{2,t+1} & \cdots & R_{N,t+1} \end{bmatrix}
  \]

Hansen and Jagannathan (1991) assume that \( R_{t+1} \) has a non-singular covariance matrix \( \Omega \). This consequently blocks the possibility of arbitrage, meaning that no asset or combination of assets offers an unconditionally risk-free positive return. They also demonstrate that the minimal volatility stochastic discount factor \( (M^*_t) \) must be a linear function of the returns on the assets, as shown in the following equation:

\[
M^*_{t+1} = \bar{M} + (R_{t+1} - E[R_{t+1}]) \beta_{M^*}
\]

(V.4)

Where: \( \bar{M} \) is the hope of all stochastic discount factor candidates, including the minimum volatility, whereas: \( E(M^*_t) = \bar{M} = E(M_{t+1}) \). The \( \beta_{M^*} \) is the vector of N linear coefficients \( \beta \), relating the deviations in return on each asset to \( M^*_t \). Except for \( \bar{M} \) all the \( M^*_t \) arguments may be calculated on the basis of historic capitals market data. Thus, before estimating \( M^*_t \) it is necessary to estimate \( \bar{M} \).

It is worthwhile recalling that \( E(M^*_t) = \bar{M} = E(M_{t+1}) \). Consequently, it is
possible to estimate $\bar{M}$ from “any” stochastic discount factor candidate $M_{t+1}$ that respects the equations (V.3) and (V.4). Assuming $U (S)$ is of the CRRA type, Equation (V.2) is:

$$M_{t+1} = \left[ \delta \left( S_{t+1} / S_t \right)^\gamma \right]$$  \hspace{1cm} (V.5)

And as it is assumed that $\delta$ is constant, this gives:

$$E(M_{t+1}) = \delta E \left[ \left( S_{t+1} / S_t \right)^\gamma \right]$$  \hspace{1cm} (V.6)

For each savings relative risk aversion level ($\gamma$) value, a historic savings growth rate series is calculated to $-\gamma$ and finally the expected value for the series is obtained: $E \left( \left( S_{t+1} / S_t \right)^\gamma \right)$. However, the problem still remains of how to determine $\delta$.

A subjective intertemporal substitution factor ($\delta$) may be obtained for each asset through rearranging the first-order condition (V.2) and Equation (V.3) in the scalar case:

$$\delta = \frac{1}{E_t \left( 1 + R_{t+1} \right) \left( S_{t+1} / S_t \right)^\gamma}$$  \hspace{1cm} (V.7)

In this case, $\delta$ and $\bar{M}$ are obtained, as an equation would be obtained for each asset where $\delta$ (and consequently $\bar{M}$) is a function of $\gamma$.

In this study, it was decided to use a matrix function in order to obtain the $M_{t+1}^*$ series, as well as the $M_{t+1}$ series, taking into consideration the rates of return in the S&P 500 Index and government papers, and the covariance matrix ($\Omega$) between them. The portfolio striving for a return identical to the risk premium will not be included, because if it were, the covariance matrix ($\Omega$) would be singular, meaning it could not be inverted ($\Omega^{-1}$). In linear algebra, this impossibility of inverting a matrix is similar to the impossibility of dividing any number by zero in scalar algebra.

The parameter $\delta$ will be estimated below, which will be unique for the entire economy, at each $\gamma$ level. It is known that the matrix version of the first-order condition of Lucas (1978), is the Equation (V.3). However, the sample mean very probably presents errors. Consequently, $\varepsilon$ represents the vector (2x1) of errors in the sample means:

$$\mathbf{1} = E \left[ (1 + \mathbf{R}_{t+1}) M_{t+1} \right] + \varepsilon$$  \hspace{1cm} (V.8)

Substituting (V.5) in (V.8) gives:

$$\mathbf{1} = \delta E \left[ (1 + \mathbf{R}_{t+1}) \left( S_{t+1} / S_t \right)^\gamma \right] + \varepsilon$$  \hspace{1cm} (V.9)

Another form of representing (V.9) would be:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \delta \begin{bmatrix} E \left[ (t+R_{S&P, t+1})(S_{t+1}/S_t)^\gamma \right] + E_{S&P} \\ E \left[ (t+R_{T-bill, t+1})(S_{t+1}/S_t)^\gamma \right] + E_{T-bill} \end{bmatrix}$$  \hspace{1cm} (V.10)

Isolating the errors vector in (V.9), and then squaring both sides of the equation gives:
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\[
\varepsilon'\varepsilon = \left( 1 - \delta E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)' \left( 1 - \delta E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right) \quad \text{(V.11)}
\]

Developing the right side of (V.11), gives:

\[
\varepsilon'\varepsilon = \varepsilon'\varepsilon + 2\delta \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)' + \delta^2 \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)' \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right) \quad \text{(V.12)}
\]

Assuming that the value of \( \delta \) is a parameter that minimizes the sample mean errors, this parameter will be estimated in a manner similar to that of the coefficients vector in a minimum squared regression. Thus, deriving the error variance (V.12) as a function of \( \delta \) and bringing it to zero, gives:

\[
\frac{\partial (\varepsilon'\varepsilon)}{\partial \delta} = -2 \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)' + 2\delta \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)' \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right) = 0
\]

\[
\text{(V.13)}
\]

In this manner, \( \delta \) is estimated through the following equation:

\[
\delta = \frac{\left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)'}{\left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)' \left( E \left[ \left( t + R_{t+1} \right) \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \right] \right)} \quad \text{(V.14)}
\]

Thus, the \( \delta \) value is a function of \( \gamma \) conditioned to the sample historic data on the rates of return and growth in savings. Once \( \delta \) is estimated, it is inserted into Equation (V.6), obtaining \( \bar{M} = E(M_{t+1}^*) = E(M_{t+1}) \). Graph V.1 demonstrates the relationships between \( \gamma, \delta, E\left( S_{t+1}/S_t \right)^{-\gamma} \) and \( \bar{M} \).
Graph V.1. Relationships between $\gamma$, $\delta$, $\bar{M}$ and $E\left[\left(S_{t+1}/S_t\right)^{-\gamma}\right]$

It is worthwhile noting that in Graph V.1, the Y axis (“normal”, on the left) refers to the $\delta$ and $\bar{M}$ values, while the Y axis (right side) refers to the $E\left[\left(S_{t+1}/S_t\right)^{-\gamma}\right]$ values. It is interesting to note that this Graph, despite a marked variation in $\delta$ and $E\left[\left(S_{t+1}/S_t\right)^{-\gamma}\right]$, $\bar{M}$ appears relatively insensitive to the $\gamma$ variations.

Through these parameters, it is also possible to obtain the entire historic series for the stochastic discount factor $(M_{t+1})$ through Equation (V.5), and consequently all its moments as a function of the relative risk aversion level ($\gamma$).

Once the methodology is defined for “constructing” historical series and the $M_{t+1}$ moments as a function of $\gamma$, it is still necessary to obtain the minimum volatility stochastic discount factor $(M_{t+1}^*)$ series. Obtained through historic or market data, this $M_{t+1}^*$ factor will be used as a validation parameter for the $M_{t+1}$ series, which was obtained through the S-CAPM approach.

According to Hansen and Jagannatan (1991), $\beta_{\Omega} = \Omega^{-1}\beta_{\Omega}$, and substituting $\beta_{\Omega}$ in Equation (V.4) gives:

$$M_{t+1}^* = \bar{M} + (R_{t+1} - E[R_{t+1}])' \Omega^{-1} (1 - \bar{M}E[R_{t+1}])$$  \hspace{1cm} (V.15)

The data in Table IV.1, meaning the historic market data, can indicate which covariance matrix ($\Omega$) is used to estimate the $M_{t+1}^*$ series:

$$\Omega = \begin{pmatrix} 0.0358 & -0.0005 \\ -0.005 & 0.0016 \end{pmatrix} \rightarrow \Omega^{-1} = \begin{pmatrix} 28.05 & 7.73 \\ 7.73 & 613.22 \end{pmatrix}$$

And the expected returns vector is:
Thus, with the historic market data and the estimate of \( \bar{M} \) (as a function of \( \gamma \)) it is possible to estimate the \( M_{t+1} \) and \( M^*_t \) “historic” series and their respective moments as a function of the relative risk aversion level (\( \gamma \)).

Graph V.2. presents the relation between the mean stochastic discount factor \[ \bar{M} = E(M^*_t) = E(M_{t+1}) \] and the standard deviations for \( M_{t+1} \) and \( M^*_t \).

Graph V.2. Relation between \( \bar{M} \), \( \sigma(M_{t+1}) \) and \( \sigma(M^*_t) \)

An analysis of Graph V.2 indicates that the \( \bar{M} \) value may not lie between an interval of approximately 0.94 and 0.97, as in this range, \( \sigma(M_{t+1}) \) is less than \( \sigma(M^*_t) \), which is by definition the volatility threshold.

In order to check these values more easily, Graph IV.7 relates \( \bar{M} \) and the difference between \( \sigma(M_{t+1}) \) and \( \sigma(M^*_t) \). Only in the region where \[ \sigma(M_{t+1}) - \sigma(M^*_t) \] is positive, may \( \bar{M} \) be located in a possible region. Thus, when interpreting Graph V.3, \( \bar{M} \) may not be expected to lie between 0.94 and 0.965.
The following Graphs demonstrate the relation between the relative risk aversion level ($\gamma$) and the standard deviations for $M_{t+1}$ and $M_{t+1}^*$. In Graph V.3, it is apparent that there are intervals where $\gamma$ does not fall within a possible region. The possible intervals for $\gamma$ are established when the corresponding standard deviations for $M_{t+1}$ are greater than the corresponding standard deviations for $M_{t+1}^*$. 

Graph V.3. Relation between $\bar{M}$ and $[\sigma(M_{t+1}) - \sigma(M_{t+1}^*)]$
Graph V.3. Relation between $\gamma$ and $\sigma(M_{t+1})$ and $\sigma(M^*_{t+1})$

Graph V.4 allows easier identification of the possible $\gamma$ Intervals. This Graph presents the relation between $\gamma$ and the difference in the $M_{t+1}$ and $M^*_{t+1}$ standard deviations.

Graph V.4. Relation between $\gamma$ $\left[\sigma(M_{t+1}) - \sigma(M^*_{t+1})\right]$

This Graph shows that $\gamma$ may not be valid in the interval between -3.75 and +1.80.
Assuming that the agents are averse to risk, $\gamma=1.80$ is the threshold of the relative risk aversion level of the savings utility function.

Thus, the validation methodology presented in this Section based on the work of Hansen and Jagannathan (1991) did not invalidate the findings presented in the previous Section. This shows that the S-CAPM may be viewed as a promising path for solving the Equity Premium Puzzle.

V. Conclusion

Although this approach seems promising, it must still be tested in other economies, especially those in which the Equity Premium Puzzle is identified. Only after empirical verification of this new approach may it be called a solution to the puzzle. Even so, it may not be the only solution.

The contribution made by this paper lies in the fact that it seeks a new starting point for deriving the utility function. Preceding works were bound to the view that the explanation must lie in consumption. However, when investigating possible differences between a modern economy and the barter economy of LUCAS (1978), the conclusion was reached that this could only be savings.

References


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1 The model derivation and notation follow Campbell *et al.* (1997).