

Nonparametric Option Pricing with Generalized Entropic Estimators ^{*}

Caio Almeida [†]Rafael Azevedo[‡]

August 28, 2014

Abstract

Pricing options in incomplete markets is a challenging task due to the existence of infinite risk-neutral measures that correctly price the underlying asset but give alternative prices for the option payoff. In this context, we analyze a large family of entropic discrepancy loss functions each implying a risk-neutral measure that takes into account specific combinations of higher moments of the underlying return process. We test the ability of these risk-neutral measures to reproduce theoretical option prices for different moneynesses and maturities when the simulated DGP for the underlying asset is given by a realistic jump-diffusion process. A specific subset of measures is identified as the best to price options under the adopted jump-diffusion model. We make use of this subset to suggest robust price intervals for options as opposed to single prices.

Keywords: Risk-Neutral Measure, Option Pricing, Nonparametric Estimation, Robustness, Minimum Contrast Estimators, Cressie Read Discrepancies.

JEL Classification Numbers: C1,C5,C6,G1.

^{*}Caio Almeida gratefully acknowledges financial support from CNPq.

[†]E-mail: calmeida@fgv.br, FGV/EPGE, Rio de Janeiro, Brazil.

[‡]E-mail: rafael.moura.a@gmail.com, FGV/EPGE, Rio de Janeiro, Brazil.

1 Introduction

Arrow-Debreu state prices are among the most important pieces of information embedded in financial instruments. Also known by state price density (SPD) or risk-neutral probability measure in a continuous-state version, they are fundamental to give prices to new (or exotic) financial instruments. The estimation of such objects has been an important topic of research studied for instance by Jarrow and Rudd (1982), Rubinstein (1994), Jackwerth and Rubinstein (1996), Stutzer (1996), Ait-Sahalia and Lo (1998), Ait-Sahalia and Duarte (2003) and more recently by Gagliardini, Gourieroux and Renault (2011), among others.

In the context of incomplete markets, when the objective is to price a derivative, we face a concrete problem in the estimation process: How to choose a risk-neutral measure from an infinite set of possible measures that are equally compatible with observed data on the underlying asset?

Some researchers privilege more robustness than precision and impose restrictions on characteristics of admissible risk-neutral measures such as limiting their variances (Cochrane and Saa-Requejo, 2000) or other quantities like Gain-loss ratios (Bernardo and Ledoit, 2000 or Cerny, 2003) . These restrictions reduce the set of admissible measures but do not determine a unique measure, therefore generating multiple prices for the derivative. Others choose from the set of admissible measures, one unique measure that minimizes a specific objective loss function like in Rubinstein (1994) or in Stutzer (1996)¹.

¹Another alternative proposed by Gagliardini, Gourieroux and Renault (2011) explicitly uses cross-sections of prices of derivatives when generalizing the GMM to reconstruct the pricing operator considering both uniform and local conditional moment restrictions. The Extended Method of Moments (XMM) accommodates simultaneously the pricing of an underlying asset whose time-series is observed, and the pricing of a few cross-sections of corresponding derivatives. XMM estimates the pricing operator based on a minimization problem subject to a set of pricing equations for the underlying asset and its derivatives where the physical probability distribution of returns is non-parametrically obtained with a kernel smoothing and a parametric SDF is estimated so as to minimize a mixed quadratic-entropic criterion.

In this paper, we are interested in pricing options with a methodology that lies in between the two possibilities described above. We analyze a comprehensive set of risk neutral measures that are derived from loss functions that belong to the Cressie-Read family of discrepancies (Cressie and Read, 1984). For each loss function one unique measure that prices the underlying asset returns is chosen. Nonetheless, looking at the family as a whole, since it contains infinite loss functions, we are able to estimate infinite measures, each one with a certain economic interpretation and having specific sensitivity to each moment of the underlying asset returns.

Almeida and Garcia (2009) showed that minimizing discrepancies on the Cressie Read family generates an infinite set of risk-neutral measures that represents specific nonlinear (hyperbolic) functions of the underlying asset returns. That means that attached to each discrepancy is a risk-neutral measure coming from an optimization problem with specific weights given to skewness, kurtosis and other higher moments of underlying returns. This family captures as particular cases the KLIC criterion adopted by Stutzer (1996) and also the Empirical Likelihood and the Euclidean divergence recently analyzed by Haley and Walker (2010) when pricing European call options.

We advocate here in favor of a robust treatment of option pricing in the spirit of Cochrane and Saa-Requejo (2000) ², by providing intervals of prices instead of point values that are usually obtained through a specific parametric model, but still being able to explicitly obtain the subset of risk-neutral measures responsible for the multiple prices given to the analyzed option. Since the real *unknown* DGP process for underlying assets in equity markets usually generates fat tailed and

²and also of Bernardo and Ledoit (2000), Cerny (2003), Bizid and Jouini (2005), and Boyle, Feng, Tian, and Wang (2008).

skewed returns, analyzing the sensitivity of risk-neutral measures to higher moments of underlying returns might provide important insights to price derivatives in incomplete markets.

Our analysis of the Cressie Read family is performed based on three DGPs (increasing in complexity) for the underlying asset price: lognormal returns in a Black and Scholes world, stochastic volatility Heston (1993) model, and the Bates (2000) jump-diffusion process. Previous papers that analyze the performance of risk neutral measures chosen by entropic methods³ have adopted either a lognormal or Heston's stochastic volatility model as the DGP for returns. However, it is now well documented that returns in equity and other markets contain jump components in addition to stochastic volatility.

Several studies estimate and test affine jump-diffusion models. Bates (2000) finds evidence against some specifications with pure stochastic volatility components or only jumps in returns and is favorable to a model that presents stochastic volatility and correlated jumps in returns and volatility (SVCJ model). Eraker *et al.* (2003), Chernov *et al.* (2003) and Eraker (2004) estimate variations of the above-mentioned SVCJ model using US stock market data and / or short panels of option prices. Broadie *et al.* (2007) complement their work by estimating the risk-neutral parameters of the SVCJ model using simultaneously derivatives and spot price data, including a much larger set of option prices in the estimation process. More recently Christoffersen, Jacobs and Ornthanalai (2012) introduce a class of discrete-time models that generalizes the SVCJ model, presenting stochastic volatility and time-varying jump intensities.

In this paper, due to its tractability and ability to fit empirical stylized facts in equity markets, we adopt the SVCJ model as the main DGP for returns of the

³See Stutzer (1996), Gray and Newman (2005) and Haley and Walker (2010).

underlying asset. In addition to capturing important characteristics of real world data, there are reliable estimates of its objective and risk-neutral parameters for equity markets (see Broadie et al. (2007)), and it presents closed-form formulas for option prices (see Duffie, Pan and Singleton (2000))⁴.

When the DGP for returns follows a log-normal distribution, we obtain (in Appendix A) an interesting analytical result stating that the best Cressie Read estimator under log-normality is a specific function of the three parameters that define risk-premia (μ, r, σ) in the Black and Scholes model. This complements the results obtained by Stutzer (1996) who analyzed numerically the properties of KLIC when pricing options under the Black and Scholes model. When the parameters for the Black and Scholes DGP are those adopted by Stutzer (1996) the risk neutral measure that best approximates the log-normal risk-neutral measure implied by the DGP does not come from the KLIC estimator but instead from an element on the Cressie Read family very close to the Empirical Likelihood estimator.

When the DGP process is given by the SVCJ model, there is no specific element within the Cressie Read family that performs best when considering options with different moneyness and time to maturity (see Table 4). However, an interesting feature of our analysis is that the best Cressie Read probability measures in terms of Mean Absolute Percentage Errors (MAPE) or Mean Percentage Errors (MPE) of options, are in a narrow range of elements of the family ($\gamma \in [-2, -0.9]$ for MAPE and $\gamma \in [-3.7, -1]$ for MPE). Those regions include the Empirical Likelihood risk neutral measure ($\gamma = -1$) found to have the best pricing performance in previous studies that did not include jumps. However, they also include other measures that

⁴Broadie and Kaya (2006) describe a method to sample from the exact model-implied distribution for prices in the SVCJ model, avoiding the bias introduced on simulations of stochastic volatility models using Euler discretization schemes. See also Duffie and Glynn (1996).

put more weights on higher moments of returns. Since pricing errors are small for a large range of option maturities and moneynesses, and since the DGP process for the underlying asset is recognized to capture well empirical features of the US equity market, we use these results to suggest a new method to price options. The idea is in the spirit of Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000) to provide intervals of prices for options as opposed to a unique price determined by a specific option pricing model⁵.

2 Risk-Neutral Measures via Canonical Valuation or Cressie-Read Discrepancies

Given a probability space (Ω, F, P) , suppose that we are interested in pricing derivatives on a certain underlying asset, whose prices $p_{\tilde{t}}$ are observed under the probability measure P . An assumption of absence of arbitrage guarantees the existence of at least one risk-neutral measure Q equivalent to P under which the discounted price of any asset is a martingale (see Duffie (2001)). In particular, considering an European Call with h days to maturity, its price at time t will be given by:

$$C = E_t^Q \left[\frac{\max(p_{t+h} - B, 0)}{(1 + r_f)^h} \right], \quad (1)$$

where E_t^Q is the conditional expectation operator under Q , B is the option strike, r_f is the risk-free rate for one day, and p_{t+h} is the price of the underlying asset at

⁵We also report in the paper, an analysis of our estimators when the DGP follows the Heston (1993) stochastic volatility model with parameters estimated in Garcia, Lewis, Pastorello and Renault (2011) based on both underlying S&P 500 returns and option prices. Due to the extremely high volatility risk premium obtained for the Heston model our estimators do not perform well under this DGP. This high estimated volatility risk premium might simply reflect a misspecification of the Heston model that does not consider the existence of jumps in returns and volatility.

time $t + h$.

Assuming stationarity and ergodicity of the underlying asset returns under P , we adopt a historical time series of its prices $\{p_{\bar{t}}\}_{\bar{t}=t-nh, t-(n-1)h, \dots, t-h, t}$ to generate a discrete version of the future price distribution. Each possible future outcome has empirical probability $\pi_k = \frac{1}{n}$ under P and is defined by $p_{t+h}^{(k)} = p_t * R_k$, where R_k is the k_{th} historical return $R_k = \frac{p_{t_0+kh}}{p_{t_0+(k-1)h}}$, $k = 1, \dots, n$.

In such a context, a sample version of the conditional expectation in Eq. (1) can be written as:

$$C = \sum_{k=1}^n \pi_k^Q \frac{\max(R_k p_t - B, 0)}{(1 + r_f)^h} \quad (2)$$

where π_k^Q is the probability of the k_{th} outcome under a discrete version of the risk-neutral measure Q .

If the number of observed states n is bigger than the number of primitives assets, the market is incomplete and in general there is an infinity of risk-neutral measures. In this setting, the pricing problem becomes how to properly choose one specific measure π_k^Q from the set of existing risk-neutral measures.

Stutzer (1996) suggested the Canonical Valuation method that consists in choosing the risk-neutral measure π^Q that is closest to the equiprobable objective measure⁶ π , by minimizing the Kulback-Leibler Information Criterion (KLIC) between π^Q and π :

$$I(\pi^Q, \pi) = \sum_{k=1}^n \pi_k^Q \log \left(\frac{\pi_k^Q}{\pi_k} \right) \quad (3)$$

One possible generalization (Walker and Halley (2010), Almeida and Garcia (2009)) is to substitute the KLIC by a more general function that captures the

⁶This is an optimal non-parametric estimator of the objective distribution given some conditions. See Bahadur et al. (1980, sec 3) .

Cressie-Read (CR) family of discrepancies:

$$CR_\gamma(\pi^Q, \pi) = \sum_{k=1}^n \pi_k \frac{\left(\frac{\pi_k^Q}{\pi_k}\right)^{\gamma+1} - 1}{\gamma(\gamma+1)} \quad (4)$$

where γ defines one function in the CR family.

Note that the KLIC is a particular case of CR:

$$\lim_{\gamma \rightarrow 0} CR_\gamma(\pi^Q, \pi) = I(\pi^Q, \pi). \quad (5)$$

Finally, making π equiprobable ($\pi_k = 1/n, \forall k$) the optimization problem becomes:

$$\pi^{Q*} = \arg \min CR_\gamma(\pi^Q, 1/n) \quad (6)$$

s.t.

$$\sum_{k=1}^n \pi_k^Q = 1 \quad (7)$$

$$\pi_k^Q > 0 \quad (8)$$

$$\frac{1}{(1+r_f)^h} \sum_{k=1}^n \pi_k^Q R_k = 1 \quad (9)$$

The first two restrictions guarantee that π^Q is a probability measure, and the last one is a pricing equation that guarantees that it is a risk-neutral measure when primitive assets are the risk-free and the underlying asset.

If the set of known prices at t includes one option with premium \tilde{C} and strike $\tilde{B} \neq B$ this information may be added to the set of restrictions:

$$\tilde{C} = \sum_{k=1}^n \pi_k^Q \frac{\max(R_k P_t - \tilde{B}, 0)}{(1+r_f)^h}. \quad (10)$$

and in this case the primitive assets include not only the risk-free and underlying asset, but also the observed option with price \tilde{C} .

2.1 The Dual Problem

The number of variables n on the optimization problem above depends on the size of the time series of returns adopted to approximate the future price distribution. In general this number is large what imposes some difficulties on the implementation of this problem. Fortunately Borwein and Lewis (1991) show that this type of convex problem can be solved in a usually much smaller dimensional dual space. In this case it is possible to show that the moment conditions (Euler Equations) that generate Lagrange Multipliers on the primal problem become the active variables λ on the following dual concave problem:

$$\hat{\lambda} = \arg \sup_{\lambda \in \Lambda} -\frac{1}{\gamma + 1} \sum_{k=1}^n (1 + \gamma \lambda (R_k - (1 + r_f)^h))^{\frac{\gamma+1}{\gamma}} \quad (11)$$

with $\Lambda = \{ \lambda \in R \mid (1 + \gamma \lambda (R_k - (1 + r_f)^h)) > 0 \text{ for all } k \}$.

The first order conditions on the problem above allow us to recover the implied risk neutral probability via the following formula:

$$\pi_k^{\gamma, Q} = \frac{\left(1 + \gamma \hat{\lambda} (R_k - (1 + r_f)^h)\right)^{1/\gamma}}{\sum_{i=1}^n \left(1 + \gamma \hat{\lambda} (R_i - (1 + r_f)^h)\right)^{1/\gamma}} \quad (12)$$

In the case of finding a risk-neutral measure that prices the underlying asset and risk-free rate, the dual problem becomes a simple one-dimensional optimization problem.

In what follows below we give a portfolio interpretation for the dual optimization problem that will be important to economically motivate the choice of some specific

implied risk neutral measures to price options.

2.1.1 Dual Problem, Utility Pricing and Representative Agent

The dual problem may be interpreted as an optimal portfolio problem. In an interesting way, if the underlying asset is the market portfolio we can relate our work to macro-finance literature⁷. One strand of this literature discusses the ability of equilibrium models to explain option prices. After the 1987's crash it seems crucial to incorporate jumps in the market portfolio process in order to accommodate the smirk in the implied volatility. Moreover a rare disaster in consumption might explain the equity premium puzzle as Barro (2006) argues convincingly with an international panel data.

There is an intimate relation between risk-neutral measure and dynamic equilibrium models. In particular, if one knows the data generating process of asset prices and the risk-neutral density, it is possible to infer the preference of a representative agent in an equilibrium model of asset prices (see Ait-Sahalia and Lo, 1998). The simplest equilibrium model we can relate to our setting, with a fixed Cressie Read discrepancy, is a two period model whose agents have the same utility function, consume only on the second period and where the market is complete. The only source of risk is the market portfolio process and each agent is endowed with a fraction of it. In this case, we have a representative agent utility demanding the whole market portfolio and nothing else.

There is a well known beautiful relationship relating the optimization problem of finding an optimal risk-neutral measure in the space of probability measures and solving a representative agent model⁸. It turns out that the latter is the objective

⁷Of particular interest here are: Riez (1988), Bates (2000), Barro (2006), Barro and Jin (2011), Backus et. al (2011) and Benzoni *et al* (2011) .

⁸See Follmer and Schweizer (2010) and references therein.

function of the dual problem of the former, i.e., the dual problem defined by Equation 11 may be interpreted as an optimal portfolio problem with HARA-type utility as shown in Almeida and Garcia (2009):

$$u(W) = -\frac{1}{1+\gamma}(1-\gamma W)^{\frac{\gamma+1}{\gamma}}, \quad (13)$$

where

$$W = W_0 \left[R_f + \hat{\lambda}(R - R_f) \right], \quad (14)$$

W_0 is the initial wealth and R_f is the gross risk free rate, R is the gross return of the risky underlying asset, with the restriction that $1 - \gamma W > 0$. The connection between the above problem and the dual problem is evident if we define $\hat{\lambda}$ as

$$\lambda = \frac{-\hat{\lambda}}{1 - \gamma W_0 R_f}, \quad (15)$$

and re-write the utility maximization problem as:

$$\sup_{\lambda \in \Lambda} E[u(W)] = \sup_{\lambda \in \Lambda} \left\{ u(W_0 r_f) E \left[(1 + \gamma \lambda (R - R_f))^{\frac{\gamma+1}{\gamma}} \right] \right\}. \quad (16)$$

3 Data Generating Processes for the Underlying Asset Dynamics

3.1 Black and Scholes Model

We first test the method adopting the B&S environment with the same parameters used in previous works (Stutzer (1996), Gray and Neumann (2005) and Haley and Walker (2010)). The objective measure drift for the stock price is given by $\mu = 10\%$, the volatility by $\sigma = 20\%$, and the risk-free rate by $r = 5\%$. The

stochastic differential equation followed by the price is:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \quad (17)$$

implying a log-normal distribution for stock prices under the probability measure P .

3.2 Heston (1993) Stochastic Volatility Model

Heston (1993) proposed an important option pricing model that included a stochastic volatility component correlated with the underlying asset Brownian source of risk. The model can be described by the following pair of Stochastic Differential Equations, first under the objective probability measure:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^S \quad (18)$$

$$dV_t = k_v (\theta_v - V_t) dt + \sigma_v \sqrt{V_t} dW_t^v \quad (19)$$

$$E [dW_t^S dW_t^v] = \rho t$$

and, under the risk neutral measure:

$$\frac{dS_t}{S_t} = r dt + \sqrt{V_t} dW_t^S \quad (20)$$

$$dV_t = k_v^Q (\theta_v^Q - V_t) dt + \sigma_v \sqrt{V_t} dW_t^v \quad (21)$$

$$E [dW_t^S dW_t^v] = \rho t$$

In the current paper, we make use of the objective and risk-neutral parameters estimated in Garcia, Lewis, Pastorello and Renault (GLPR, 2011) adopting S&P 500 daily data from January 1996 to December 2005, who find a positive large volatility risk premium, when defined by $k_v - k_v^Q$.

Since in GLPR (2011) they don't report the estimated drift μ nor the risk-free rate, we arbitrarily set the drift to 10% and the risk-free rate to the year 2000 average of the 1-year T-Bill. We also set the initial value V_0 of the volatility equal to its long-term average: $V_0 = \theta_v$. In order to sample from this model, we follow the method by Broadie and Kaya (2006).

3.3 Stochastic Volatility Models with Jumps

As mentioned before, we follow Bates (2000) adopting the stochastic volatility model with correlated jumps as the DGP process for equity returns. It is defined by two stochastic differential equations respectively for the price and volatility of the underlying asset. The jumps in the price and in the volatility process happen at the same time and are therefore perfectly correlated. The equity index price, S_t , and its spot variance, V_t , solve:

$$dS_t = S_t \mu dt + S_t \sqrt{V_t} dW_t^s + d \left(\sum_{n=1}^{N_t} S_{\tau_n^-} [e^{Z_n^s} - 1] \right) \quad (22)$$

$$dV_t = \kappa_v (\theta_v - V_t) dt + \sigma_v \sqrt{V_t} dW_t^v + d \left(\sum_{n=1}^{N_t} Z_n^v \right) \quad (23)$$

$$\mu = r_t - \delta_t + \gamma_t - \bar{\mu}_s \lambda \quad (24)$$

$$\bar{\mu} = \exp(\mu_s + \sigma_s^2/2) - 1 \quad (25)$$

where (dW_t^s, dW_t^v) is a bi-dimensional Brownian motion with $E[dW_t^s dW_t^v] = \rho t$,

N_t is the number of jumps until time t described as a Poisson process with intensity λ ; Z_n^v is the $n_t h$ jump in volatility with an exponential distribution with mean μ_v ; Z_n^s is associated to the $n_t h$ jump in price with a normal distribution conditional on Z_n^v with mean $(\mu_s + \rho_s Z_n^v)$ and variance σ_s^2 ; τ_n is the time of $n_t h$ jump, r is the risk-free rate, δ is the dividend yield and γ is the equity premium.

We choose this model mainly because it acknowledges several characteristics found in the real world empirical stylized facts (see Backus, Chernov and Martin, 2012) and because there are reliable estimates of its parameters for the equity market⁹.

3.3.1 Chosen Parameters

Note that the market is incomplete whenever there is stochastic volatility and/or jumps with only one underlying and one risk-free asset. A direct consequence is the existence of an infinity of risk-neutral measures consistent with the prices of such assets.¹⁰ The usual way to deal with this issue is to parameterize the possible changes of measure and make specific assumptions about the distributions of jumps. In this context, we follow Duffie *et al.* (2000) and Broadie *et al.* (2007) by considering the following stochastic differential equations under the risk-neutral measure:

$$dS_t = S_t \mu^Q dt + S_t \sqrt{V_t} dW_t^s(Q) + d \left(\sum_{n=1}^{N_t(Q)} S_{\tau_n^-} [e^{Z_n^s(Q)} - 1] \right) \quad (26)$$

$$dV_t = \kappa_v^Q (\theta_v - V_t) dt + \sigma_v \sqrt{V_t} dW_t^v(Q) + d \left(\sum_{n=1}^{N_t} Z_n^v(Q) \right) \quad (27)$$

⁹See Eraker *et al.* (2003), Chernov *et al.* (2003), Eraker (2004), and Broadie *et al.* (2007).

¹⁰In fact, the Girsanov theorem imposes very weak conditions for the jump-distribution change of measure. See for instance the appendix in Pan (2002).

$$\mu^Q = r_t - \delta_t - \bar{\mu}_s^Q \lambda^Q \quad (28)$$

$$E[dW_t^s(Q)dW_t^v(Q)] = \rho t \quad (29)$$

The objective measure's simulations uses the objective measure parameters from Table 1. The "correct" option prices are calculated with the formula obtained by Duffie *et al.* (2000) using the risk-neutral parameters (reported in the same table) and SDEs that appear in Equations (26) to (29). Those parameters are borrowed from estimations in Eraker *et al.* (2003) for the objective measure and from Broadie *et al.* (2007) for the risk-neutral measure. The risk-free rate r is the one year average of the 1-year T-Bill during the year 2000 and the initial volatility is $\sqrt{V_0} = 0.19$.

The absolute continuity requirement implies that some parameters (or combination of parameters) should be the same under both measures. In particular in our case, this is true for σ_v , ρ and the product $\kappa_v \theta_v$. We also consider that the arrival intensity is a constant, that $Z_n^s(Q)$ has a normal distribution $N(\mu_s^Q, (\sigma_s^Q)^2)$ and that $Z_n^v(Q)$ has an exponential distribution with mean μ_v^Q .

3.3.2 Exact Simulation

It is well known that approximating a continuous time price process by a discrete time process may generate bias on the final price. In general, this bias decreases as the number of steps increases. For the Euler scheme, under certain conditions described by Kloeden and Platen (1992), there is a first order convergence rate. Nonetheless, stochastic volatility processes do not satisfies such conditions. In fact, Broadie and Kaya (2006) find that the bias may be very large in some cases even if a large number of steps are used.

For this reason, we work with exact sampling from the SVCJ by using the

method described by Broadie and Kaya (2006). Notice that for the times between jump arrivals, the process behaves exactly as the Heston (1993) model. Therefore, after sampling the jump times and sizes $(\tau_n, Z_n^s \text{ and } Z_n^v)$, there is only the additional necessity of simulating a Heston-type model. To see how this can be done, write the stock price and the variance at t as:

$$S_t = S_u \exp \left[\mu(t - r) - \frac{1}{2} \int_u^t V_s ds + \rho \int_u^t \sqrt{V_s} dW_s^v + \sqrt{1 - \rho^2} \int_u^t \sqrt{V_s} dW_s^2 \right], \quad (30)$$

$$V_t = V_u + \kappa_v \theta_v(t - u) - \kappa_v \int_u^t V_s ds + \sigma_v \int_u^t \sqrt{V_s} dW_s^v, \quad (31)$$

where dW_t^v and dW_t^2 are independent and dW_t^s is decomposed as:

$$dW_t^s = \rho dW_t^v + \sqrt{1 - \rho^2} dW_t^2. \quad (32)$$

Cox *et al.*(1985) show that V_t conditional on V_u has a non-central chi-squared distribution. Broadie and Kaya (2006) find the Laplace transform of the distribution of $\int_u^t V_s ds$. The inversion of the Laplace transform can be performed in an optimized way by using the numerical integration method described by Abate and Whitt(1992). After sampling those two quantities, it is possible to obtain $\int_u^t \sqrt{V_s} dW_s^v$. Note that it is easy to sample S_t as its distribution is lognormal conditional on $\int_u^t \sqrt{V_s} dW_s^v$, $\int_u^t V_s ds$ and on V_u . For more details see Broadie and Kaya (2006).

4 Results

This section reports results concerning the applicability of the Generalized Entropic Estimators to option pricing in the Black-Scholes-Merton (B&S), Heston (1993), and SVCJ models. The SVCJ model is analyzed due to its strong ability to fit stylized facts of the US equity market while the B&S model allows us to obtain analytical properties of the Generalized Entropic Estimators that are helpful in interpreting results from previous studies. Adopting the Heston model is a way of shutting down jumps effects and concentrating on stochastic volatility as the main source of risk premium in the market.

Each model provides a theoretical option price¹¹ to be benchmarked by our non-parametric Cressie Read risk-neutral measures. A Monte Carlo study generates different realizations for the path of the underlying asset allowing us to approximate the probability distributions of the option pricing errors. For each DGP process we analyze two statistics based on the error probability distributions: The mean percentage pricing error (MPE) and the mean absolute percentage pricing error (MAPE).

Now, consider pricing an European call option on a company X with maturity h , assuming that a history of the stock prices represent the only available information. In our study, the time series of prices (or returns) will be simulated from each adopted DGP.

For a given model (or DGP), we draw a time series of returns with fixed length (in our case, 200 monthly returns) from the model-implied return distribution. Using those returns, we use the Cressie Read method with different values of γ indexing the $CR_\gamma(\cdot)$ function to price the option. For each of 71 equally spaced

¹¹The premiums are given as the no-arbitrage price implied by the risk-neutral parameters fixed for each model (B&S, Heston, or SVCJ model).

γ 's ranging from -5 to 2 we obtain one option price¹².

The procedure above is repeated 5000 times in order to obtain a distribution for the pricing errors and calculate the MPE and MAPE reported.

4.1 Black and Scholes Model

Figure 1 superimposes 20 graphs of MPE versus γ for the entropic method applied to the B&S model. Each graph corresponds to a call option with different maturity and/or moneyness. An apparently striking feature of this picture is that all graphs cross the horizontal line around $\gamma \approx -0.8$. This suggests that in principle the true risk-neutral measure under the specific log-normal DGP might be precisely estimated using our nonparametric family for some γ close to -0.8 . We provide theoretical results (see Appendix A) showing that in fact if we fix the Cressie Read parameter γ equal to:

$$\gamma^* = -\frac{\sigma^2}{\mu - r} \quad (33)$$

the corresponding Cressie Read risk-neutral measure coincides with the risk-neutral measure implied by the B&S model. From a theoretical point of view, if prices satisfies the B&S log-normal dynamics the best Generalized Entropic Estimator will be the one with γ^* from Equation (33). Substituting the parameters used in our simulations at Equation (33) we obtain $\gamma^* = -0.8$.

Note that both numerical and theoretical results indicate that the most appropriate γ under the B&S DGP is -0.8 . This is in accordance to results obtained in previous papers¹³ which have found a negative significant MPE when using $\gamma = 0$

¹²There are some difficulties to obtain the implied risk-neutral measure for some positive γ 's. For instance, the solution of the optimization problem defined by Equations (6)-(9) may not exist when we demand that $\pi_k^Q > 0, \forall k$ for some positive γ 's.

¹³See Table 1 on page 994 in Haley and Walker (2010) and Table 1 on page 7 in Gray and Newman (2005).

or $\gamma = 1$ contrasting with a positive close to zero MPE when using $\gamma = -1$ ¹⁴.

4.2 Heston Model

It is also possible to draw more informative graphs of MPE (or MAPE) versus γ 's for call options with a given maturity and moneyness. These graphs are shown in Figures 2 to 9 for the B&S, Heston and SVCJ models. The overall pattern of the MPE (depending on γ) is a function with negative slope, which is flatter for short-maturity options.

For the B&S and SVCJ models, it crosses the horizontal axis ($MPE = 0$) for γ 's within the interval $(-3.7, -.07)$. Nonetheless, for the Heston model with a positive and high volatility risk premium as reported by GLPR (2011), the MPE does not cross the horizontal axis for any $\gamma \in [-5, 0]$ (see Figure 4). In this case, the MPE is negative for all members of the Cressie Read family, and the minimum MAPE is achieved for $\gamma = -5$ for most maturities and moneynesses considered. It is interesting to note that the CR estimator with $\gamma = -5$ is the one that puts larger weights to extreme events. One possible interpretation of these negative results for the Cressie Read estimators is that the volatility risk premium within the Heston model as estimated by GLRP (2011) is too high to be captured by a risk neutral measure that is only a function of the historical distribution of the underlying asset returns. In such case, information on at least one option would be needed to identify this premium within the risk-neutral measure implied by the CR estimators.

In unreported tests available upon request, we varied the volatility risk premium from zero to the estimated value in GLPR to better understand the relation

¹⁴Note that $\gamma = -1$, $\gamma = 0$, and $\gamma = 1$ correspond, respectively, to Empirical Likelihood, Euclidean Divergence and KLIC.

between MPE, MAPE with the volatility premium. The main effect obtained with changes in the premium was a vertical translation in the MPE graphs. As a result, the MPE graphs end up crossing the horizontal axis (achieving zero) for any value of $\gamma \in [-5, 0]$ depending on the volatility risk premium. As expected, the value of γ for which the MAPE achieves its lowest value also changes as a function of volatility risk premium.¹⁵ This result illustrates well that the a risk-neutral measure can't be identified using solely information on the underlying asset returns under the objective measure. In particular, when considering the Heston model, there is no way to identify the volatility risk premium $k_v - k_v^Q$ using only return data from the underlying asset.

4.3 SVCJ Model

As pointed out in section 3 the SVCJ is a fairly parsimonious model that accommodates several important characteristics of equity prices. It accommodates a realistic process for volatility and a simultaneous jump in price and volatility. We proceed to simulations using the parameter estimates in Eraker *et al.* (2003) and Broadie *et al.* (2007) and price a set of call options with different maturities and moneynesses. The price errors are obtained comparing the prices given by our Generalized Entropic Estimators to the theoretical prices of the SVCJ model given in closed-form based on Duffie *et al.* (2000) techniques.

Unlike the B&S model, the simulations suggest that there is no clear element of the CR family function for which the MPE is zero. Also unlike the Heston model, under the SVCJ model with parameters based on the option pricing literature as

¹⁵Note that most volatility risk premium values used in this test are within one standard error of the estimated premium in GLRP. The estimated volatility risk premium $k_v - k_v^Q$ was 4.08 with a standard error of 2.88. Note that although we didn't test for $k_v - k_v^Q > 4.08$, the expected effect would be an increase of the mean error by pushing the MPE graphs downwards.

reported in section 3.3.1, the Cressie Read entropic estimators perform well with MPEs achieving zero for most moneynesses and maturities.

Table 5 indicates that the best estimator varies with maturity. For instance, the gammas that minimize MPE are close to $\gamma \approx -3.2$ and $\gamma \approx -1.1$ for maturities equal to 1-month and 12-months respectively¹⁶.

Table 5 indicates that the lowest MAPE is inside a narrower interval $\gamma \in (-2.1, -0.9)$ than the one obtained with MPE, apart from the two cheapest options. Based on the current results under the SVCJ model, we will consider obtaining intervals for option prices based on CR estimators within an interval of γ 's.

5 Robust Price intervals

We test the method for different models with typical (or estimated) parameters from U.S. stock market. The performance in terms of MPE is good and relatively similar for the B&S and SVCJ models. In general, the average pricing error is zero for some discrepancy function within the Cressie Read family in almost all cases. The only exception is found on the SVCJ model for the deep out-the-money option with short maturity.¹⁷

The discrepancy function with best performance in general depends on the DGP, moneyness and option maturity. We show that the best performance γ for the B&S model depends on the model parameters (but not on maturity). On the other hand, simulations suggest that the optimal γ varies with maturity and moneyness for the SVCJ model.

¹⁶For $S/B = 1.125$ and 1-month maturity MPE is zero for $\gamma \approx -3.7$. Nonetheless the graph MPE versus γ has a very small slope and therefore is more prone to statistical error linked to Monte Carlos studies. See Figures 2 to 7 for more details.

¹⁷The method is less successful for the Heston model due to the high volatility risk premium reported in the literature and to the fact that the Heston model doesn't consider the existence of jumps.

Now, in the spirit of Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000) we propose an alternative way to look at this family of discrepancies. Instead of trying to obtain an optimal dependence of γ on pricing errors based on an specific option pricing model (like B&S, Heston or SVCJ), we suggest using this family to provide intervals of prices for options. These intervals would be obtained by applying the method to an interval of γ 's. This would correspond to giving option prices compatible with a set of different HARA utility functions. In particular, this method could be useful for pricing options in illiquid markets or over-the-counter derivatives.

5.1 No-Arbitrage Price Interval

The no-arbitrage price interval only requires that there are no-arbitrage opportunities in the market. The no-arbitrage price interval at time t for a European call option with maturity $t + h$ and strike B may be defined by the relations:

$$C \leq S_t, \tag{34}$$

$$C \geq \max \left\{ 0, S_{t+h} - \frac{B}{(1+r^f)^h} \right\} \tag{35}$$

where $C = E[m * \max\{0, S_{t+h} - B\}]$ is the call premium and m is a Stochastic Discount Factor (SDF hereafter) that prices the underlying and risk-free assets.

Note that the call option price can not be higher than the current stock price otherwise one could sell the option, buy the stock to hedge generating an immediate arbitrage. Similarly, if the price is smaller than $S_{t+h} - \frac{B}{(1+r^f)^h}$ one could buy the option sell the stock and lend $\frac{B}{(1+r^f)^h}$ generating again an arbitrage¹⁸.

As a matter of convenience we introduce the concept of SDF here, keeping in

¹⁸See Hull (2011) for a detailed explanation.

mind that a given SDF is equivalent to the following risk-neutral measure¹⁹:

$$m_i = \frac{1}{(1 + r^f)^h} \frac{\pi_i^Q}{\pi_i} \quad (36)$$

where m_i is the SDF value at state i .

Note that despite being robust, such price interval isn't very useful in practice since it is too wide. For this reason, some authors have tried to define tighter intervals guided by some set of economic arguments.

5.2 Good Deals and Gain-Loss Ratio

Cochrane and Saa-Requejo (2000) define a price interval that rules out too good opportunities. By too good they mean strategies that achieve very high Sharpe ratios. They define an upper bound for the Sharpe ratio and find a price interval that is consistent with this bound.

In order to implement their idea, they use the well-known Hansen and Jagannathan (1991) bounds to relate the Sharpe ratio of an arbitrary strategy to the variance of an arbitrary admissible SDF:

$$\frac{E[R - (1 + r^f)]}{\sigma(R)} \leq \left\{ \frac{\sigma(m)}{E[m]} \right\} \quad (37)$$

where $\sigma(\cdot)$ is the standard deviation $\sigma(R) = \sqrt{E[(R - E[R])^2]}$, R is the return on a strategy and m is any admissible SDF that prices R .

Suppose we want to price an European call option with maturity $t+h$ and strike B , and that we only know the underlying asset price at time t and the risk-free rate r^f . In this case, their price interval is defined by $[\underline{C}, \overline{C}]$ with:

¹⁹Assuming for the sake of simplicity that the economy has discrete states of nature.

$$\overline{C} = \max_m E[mX]$$

$$\underline{C} = \min_m E[mX] \quad (38)$$

where m satisfies the following conditions for both optimization problems:

$$m > 0 \quad (39)$$

$$E[mS_{t+h}] = S_t \quad (40)$$

$$E[m] = \frac{1}{(1 + r^f)^h} \quad (41)$$

$$\sigma(m) \leq \frac{H}{(1 + r^f)^h}. \quad (42)$$

where H is the maximum Sharpe ratio, $X = \max\{0, S_{t+h} - B\}$ is the call payoff and $\sigma(m) = \sqrt{E[(m - E[m])^2]}$.

Similarly, Bernardo and Ledoit (2000) rule out too good trading opportunities by using a different notion for good opportunity. They define the Gain-Loss ratio with respect to a SDF m^* (the benchmark SDF) for an excess payoff (a payoff whose price is zero). The SDF m^* does not need to price correctly the assets and it is used only as a way to define this attractiveness measure. For an excess payoff X^e , the Gain-Loss ratio with respect to m^* is

$$L = \frac{E[m^*(X^e)^+]}{E[m^*(X^e)^-]} \quad (43)$$

where $(X^e)^+ = \max\{X^e, 0\}$ is the positive part of the payoff X^e (the gain) and $(X^e)^- = \max\{-X^e, 0\}$ is the negative part of the payoff X^e (the loss). Note that

if m^* prices correctly X^e we have that $L = 1$ because

$$\begin{aligned} E[m^* X^e] &= 0 \\ E[m^* (X^e)^+ - m^* (X^e)^-] &= 0 \\ E[m^* (X^e)^+] &= E[m^* (X^e)^-] \end{aligned}$$

leading to

$$L = \frac{E[m^* (X^e)^+]}{E[m^* (X^e)^-]} = 1. \quad (44)$$

In order to implement their method, the authors prove a duality result that implies the following bound:

$$\frac{E[m^* (X^e)^+]}{E[m^* (X^e)^-]} \leq \frac{\sup_i \left(\frac{m_i}{m_i^*} \right)}{\inf_i \left(\frac{m_i}{m_i^*} \right)} \quad (45)$$

where m is any admissible SDF, that is, it correctly prices X^e . Note that it defines a type of variational measure. For instance, if m^* is constant the right hand side of the above equation would be the ratio of the maximum value to the minimum value of the SDF m :

$$\frac{\sup_i (m_i)}{\inf_i (m_i)} \text{ for } m^* \text{ constant.} \quad (46)$$

In order to price any new asset, they define an upper bound to the Gain-Loss ratio and implement this idea using the inequality above. For instance, suppose again that we want to price an European call option with maturity $t+h$ and strike B , and we only know the underlying asset price at t and the risk-free rate r^f . Then we would have:

$$\overline{C} = \max_m E[mX] \quad (47)$$

$$\underline{C} = \min_m E[mX] \quad (48)$$

where m satisfies the following conditions for both optimization problems:

$$m > 0 \quad (49)$$

$$E[mS_{t+h}] = S_t \quad (50)$$

$$E[m] = \frac{1}{(1 + r^f)^h} \quad (51)$$

$$\frac{\sup \left(\frac{m_i}{m_i^*} \right)}{\inf \left(\frac{m_i}{m_i^*} \right)} \leq \bar{L} \quad (52)$$

where \bar{L} is the maximum Gain-Loss ratio and $X = \max \{0, S_{t+h} - B\}$ is the Call payoff.

Note that the difference between the two methods for the present application appears specifically in Equations (42) and (52).

5.3 Robust Price Interval for $\gamma \in [-4, -0.5]$

In this section, based on the good performance of the Cressie Read implied risk-neutral measures under the SVCJ Monte Carlo experiment, we suggest intervals of prices for options. Our intervals are different from Cochrane and Saa-Requejo (2000) or Bernardo and Ledoit (2000). While they restrict the family of SDFs by fixing limits to a specific criterion like variance or Gain-Loss ratio, we consider several different discrepancies each generating an SDF compatible with a dual HARA utility function to obtain our price intervals.

Our intervals are robust in the sense that they do not rely on an specific option pricing model that would give a unique price for any option.

Table A shows price intervals for option prices with different maturities and moneynesses. This table contains the price interval for a call option under the SVCJ model given by the Cressie Read method when applied with $\gamma \in [-4, -0.5]$. The underlying asset price at t has a fixed value $S = 100$, for all cells but the strike B and maturity h change. Each cell shows the interval $(C_{\gamma=-0.5}, C_{\gamma=-4})$ along with the theoretical model-implied option price given below the interval. Objective and risk-neutral parameters adopted for the SVCJ model appear in Table 1. The value of C_γ is calculated as the average option price obtained using our method when applied to 5000 different realization of the SVCJ process. The theoretical option price is obtained by the average of all 5000 model-implied prices obtained for each realization of the SVCJ model.

Note that comparing theoretical prices with price intervals our proposed pricing method works well for most maturities and moneynesses with theoretical prices appearing within the pricing intervals. There are only a few exceptions that worth mentioning. It appears to be a hard job for the nonparametric Cressie Read family to price too short-maturity call options, in especial the short-maturity ones that are deep out-of-the-money. This can probably be explained by an inability to put high probability mass in too extreme positive events.

Here, we are still in a preliminary phase and need a more careful analysis before reaching any further conclusion... We intend to provide further tests with put options and also use those intervals to analyze real prices of options written on the S&P 500 index.

6 Conclusion

In this work we study the performance of a non-parametric option pricing method when the underlying asset follows a realistic jump-diffusion model. We try to find the risk-neutral measure within a certain class of entropic measures that best approximates, from an option pricing perspective, a given DGP process for the underlying asset.

We simplify a minimization problem in the space of risk-neutral measures by solving an optimal portfolio problem on the dual space of returns of the underlying process. By simulating the jump-diffusion process proposed by Bates (2000) with parameters following recent studies in the option pricing literature (Eraker *et. al* (2003), and in Broadie *et. al* (2007)), we show that the most appropriate entropic risk-neutral measures are very sensitive to higher moments of returns in the dual space (γ 's ranging between -3 and -1).

We conclude by proposing price intervals for option prices that are obtained by focusing on an interval of γ 's that parameterize our entropic family. Such intervals are compatible with giving option prices based on a set of HARA utility functions whose average risk-aversion is parameterized by the γ parameter.

From a pricing perspective, our results might be used to provide robust price intervals for derivatives in illiquid and over the counter markets.

A Nonparametric Pricing Method Applied to B&S Model: An Exact Estimation

In this appendix, we show that the nonparametric method defined in Section 2 provides the correct option prices when applied to the B&S model, if we choose

a specific discrepancy function within the Cressie Read family. In particular, we show that it is possible to obtain the model implied risk-neutral distribution from the B&S model when we solve the optimization problem (equations (6)-(9)) with the following γ :

$$\gamma^* = -\frac{\sigma^2}{\mu - r}. \quad (53)$$

That is, the Cressie Read discrepancy function that reproduces the B&S implied risk-neutral measure depends on the parameters of the B&S model but neither on maturity nor moneyness of any option being priced²⁰.

To solve the above-mentioned problem, we consider the optimization problem of our entropic nonparametric method applied to the continuous distribution of returns in B&S model, and show that the Radon-Nikodym derivative obtained by the optimization problem is the same as the Radon-Nikodym derivative implied by the B&S model when $\gamma = \gamma^*$. This implies that when the method is applied to a finite sample with γ^* we obtain an approximation for the correct Radon-Nikodym derivative.

We begin by writing the Radon-Nikodym derivative in the B&S model as a function of the returns. Then we write the optimization solution of our nonparametric method in a suitable way and finally note that the B&S Radon-Nikodym derivative is the solution for the nonparametric method if γ^* is used²¹.

²⁰This is an interesting result since in general for more complex DGP processes the best Cressie Read estimator will depend on option maturity and or moneyness as shown in the paper for the SVCJ model.

²¹For the B&S model, the method using only one pricing equation restriction may be regarded as an Esscher transform. Option pricing with Esscher transform are studied in Gerber and Shiu (1994) along with several others authors discussing it. In particular Y. Yao's response (Gerber and Shiu (1994: page 168-173) shows that the Esscher transform obtains the risk-neutral measure implied by the Black-Scholes model using tools from martingale theory. The arguments used here are similar to Yao's.

In the Black and Scholes model, we have under the objective measure:

$$\ln \left(\frac{S_t}{S_u} \right) = \left(\mu - \frac{1}{2} \sigma^2 \right) (t - u) + \sigma (W_t - W_u). \quad (54)$$

where μ is the continuous expect rate of return, σ is the volatility and W_t is the Wiener process. Rewriting Equation (54) we get:

$$W_t - W_u = \frac{1}{\sigma} \left[\ln \left(\frac{S_t}{S_u} \right) - \left(\mu - \frac{1}{2} \sigma^2 \right) (t - u) \right] \quad (55)$$

Defining the gross return between t and u as

$$R_{u,t} = \frac{S_t}{S_u}, \quad (56)$$

we have

$$W_t - W_u = \frac{1}{\sigma} \left[\ln (R_{u,t}) - \left(\mu - \frac{1}{2} \sigma^2 \right) (t - u) \right]. \quad (57)$$

In order to change to the risk-neutral measure guaranteed to exist by no-arbitrage conditions one may apply Girsanov's theorem. In this case, the Radon-Nikodym derivative is:

$$Z(t) = \exp \left\{ -\theta (W_t - W_0) - \frac{1}{2} \theta^2 t \right\}, \quad (58)$$

where

$$\theta = \frac{\mu - r}{\sigma}, \quad (59)$$

and r is the risk-free rate. We can write $Z(t)$ as a function of returns:

$$Z(t) = \exp \left\{ -\theta \frac{1}{\sigma} \left[\ln (R_{0,t}) - \left(\mu - \frac{1}{2} \sigma^2 \right) t \right] - \frac{1}{2} \theta^2 t \right\} \quad (60)$$

$$Z(t) = \exp \left\{ \ln \left((R_{0,t})^{-\frac{\theta}{\sigma}} \right) + \frac{\theta}{\sigma} \left(\mu - \frac{1}{2} \sigma^2 \right) t - \frac{1}{2} \theta^2 t \right\}$$

$$Z(t) = (R_{0,t})^{-\frac{\theta}{\sigma}} \exp \{At\}. \quad (61)$$

where

$$A = \frac{\theta \mu}{\sigma} - \frac{1}{2} \sigma \theta - \frac{1}{2} \theta^2. \quad (62)$$

The fact that the Radon-Nikodym derivative on the Girsanov theorem is a martingale (under the objective measure) with $Z_0 = 1$ implies that:

$$E[Z_t] = 1.$$

directly giving:

$$E \left[(R_{0,t})^{-\frac{\theta}{\sigma}} \right] = \exp \{-At\} \quad (63)$$

where $E[\cdot]$ is the expectation under the objective probability measure. Moreover, we have by the properties of Radon-Nikodym derivative and risk-neutral measures

$$E[R_{0,t}Z(t)] = \tilde{E}[R_{0,t}] = e^{-rt}. \quad (64)$$

The last two relations will be useful to obtain the equivalence between the B&S implied risk-neutral measure and the one implied by the Cressie Read method with γ^* .

Now, from the entropic Cressie Read point of view, given an option with time to maturity t define $R = R_{0,t}$. The solution of the Cressie Read optimization problem implied by the nonparametric method is given by the following Radon-Nikodym

derivative (see Almeida and Garcia (2009)):

$$\frac{dQ}{dP} = \frac{(1 + \gamma\lambda(R - R_f))^{\frac{1}{\gamma}}}{E \left[(1 + \gamma\lambda(R - R_f))^{\frac{1}{\gamma}} \right]}. \quad (65)$$

Under the Black and Scholes model, R is lognormal (see equation (54)). Since the risk-neutral measure has to price the underlying asset, the optimization problem has the following restriction

$$\frac{1}{R_f} E^Q [R] = 1,$$

or, equivalently,

$$\frac{1}{R_f} E \left[\frac{dQ}{dP} R \right] = 1, \quad (66)$$

along with $1 + \gamma\lambda(R - R_f) > 0$ for all states. If there is a λ such that Equation (66) holds and $1 + \gamma\lambda(R - R_f) > 0$ $\frac{dQ}{dP}$ from Equation (65) will be the solution. Moreover, this solution will be unique because the dual problem is strictly concave.

Now, define (implicitly) $\hat{\lambda}$ as

$$\lambda = \frac{\hat{\lambda}}{\gamma R_f}.$$

to obtain

$$\begin{aligned} \frac{dQ}{dP} &= \frac{\left(1 + \gamma \frac{\hat{\lambda}}{\gamma R_f} (R - R_f)\right)^{\frac{1}{\gamma}}}{E \left[\left(1 + \gamma \frac{\hat{\lambda}}{\gamma R_f} (R - R_f)\right)^{\frac{1}{\gamma}} \right]} \\ \frac{dQ}{dP} &= \frac{\left(\frac{1}{R_f}\right)^{\frac{1}{\gamma}} \left(R_f + \hat{\lambda} (R - R_f)\right)^{\frac{1}{\gamma}}}{\left(\frac{1}{R_f}\right)^{\frac{1}{\gamma}} E \left[\left(R_f + \hat{\lambda} (R - R_f)\right)^{\frac{1}{\gamma}} \right]} \end{aligned}$$

$$\frac{dQ}{dP} = \frac{\left(R_f(1 - \hat{\lambda}) + \hat{\lambda}R\right)^{\frac{1}{\gamma}}}{E\left[\left(R_f(1 - \hat{\lambda}) + \hat{\lambda}R\right)^{\frac{1}{\gamma}}\right]}. \quad (67)$$

Finally, the trick is to make the ansatz $\hat{\lambda} = 1$ that will eliminate the risk-free asset R_f from the Radon-Nikodym measure.

$$\frac{dQ}{dP} = \frac{R^{\frac{1}{\gamma}}}{E\left[R^{\frac{1}{\gamma}}\right]}$$

Now, in order to find the correct γ , just compare the above $\frac{dQ}{dP}$ with $Z(t)$. This suggest the choice $1/\gamma = -\theta/\sigma$ that makes $\frac{dQ}{dP} = Z(t)$, where t is the time to maturity and $E\left[R^{\frac{1}{\gamma}}\right] = \exp\{-At\}$ by equation (63). In order to show that this is the solution, it is necessary to verify that $1 + \gamma\lambda(R - R_f) > 0$ and that Equation (66) holds. Indeed, $1 + \gamma\lambda(R - R_f) = R/R_f > 0$ a.s. because R is lognormal and noting that $R_f = e^{-rt}$ we have

$$\frac{1}{R_f} E\left[\frac{dQ}{dP} R\right] = \frac{1}{e^{-rt}} E[Z(t)R] = 1,$$

as expected.

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Table 1: Objective (upper line) and risk-neutral (lower line) measure parameters for SVCJ model.

μ or r	κ_v	θ_v	σ_v	ρ	λ	μ_s	σ_s	μ_v	ρ_s
0.1386	6.5520	0.0136	0.4	-0.48	1.5120	-0.0263	0.0289	0.0373	-0.6
0.0593	6.5520	0.0136	0.4	-0.48	1.5120	-0.0725	0.0289	0.1333	0.0

Table 2: Objective (upper line) and risk-neutrals (other two lines) measure parameters for Heston model. The middle and the lower line correspond to, respectively, the estimated risk neutral measure in Garcia et al. (2011) and no volatility risk premium.

μ or r	k_v	θ_v	σ_v	ρ
0.10	6.48	0.0203	0.4464	-0.215
0.0593	2.4	0.0547	0.4464	-0.215
0.0593	6.48	0.0203	0.4464	-0.215

Table 3: Optimal Cressie-Read discrepancy function for B&S model.

This table contains the Cressie-Read discrepancy function which attains the minimum error for the B&S model. The first panel displays the γ in which the method has zero mean percentage error (MPE). The second panel displays the γ in which the method has the lowest mean absolute percentage error (MAPE). Note that the MPE is zero for $\gamma = -0.9$ for most entries. In matter of fact the MPE is almost zero for $\gamma = -0.9$ and for $\gamma = -0.8$. For the most entries the MAPE is almost the same for some γ close to the lowest one (sometimes it is equal to the fifth decimal place). Each cell is associated with an European Option Call with a different combination of moneyness and maturity. The index γ defines the Cressie-Read function through the function $CR_\gamma(\cdot)$ - see equation (4). Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions.

γ with MPE equal to zero	Maturity (year)			
	1/12	1/4	1/2	1
S/B = 0.90	-0.7	-0.8	-0.8	-0.9
S/B = 0.93	-0.9	-0.9	-0.8	-0.9
S/B = 1.00	-0.9	-0.9	-0.8	-0.9
S/B = 1.03	-0.9	-0.9	-0.9	-0.9
S/B = 1.125	-1.0	-0.9	-0.9	-0.9
γ with minimum MAPE	Maturity (year)			
	1/12	1/4	1/2	1
S/B = 0.90	.	-0.5	-0.7	-0.8
S/B = 0.93	-0.5	-0.8	-0.7	-0.8
S/B = 1.00	-0.7	-0.7	-0.8	-0.8
S/B = 1.03	-0.3	-0.7	-0.8	-0.9
S/B = 1.125	0.5	-0.3	-0.5	-0.7

Table 4: Optimal Cressie-Read discrepancy function for Heston model.

This table contains the Cressie-Read discrepancy function which attains the minimum error for the Heston model. In the cell, the left and right number is associated with the estimated and zero volatility risk premium, respectively. Each cell is associated with an European Option Call with a different combination of moneyness and maturity. The first panel displays the γ in which the method has zero mean percentage error (MPE). The second panel displays the γ in which the method has the lowest mean absolute percentage error (MAPE). The index γ defines the Cressie-Read function through the function $CR_\gamma(\cdot)$ - see equation (4). Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The entries with n/a means that no γ in the range $(-5, 2)$ has zero MPE.

γ with MPE equal to zero	Maturity (year)			
	1/12	1/4	1/2	1
S/B = 0.90	n/a -1.8	n/a -0.7	n/a 0	n/a n/a
S/B = 0.93	n/a 0	n/a n/a	n/a 0	n/a n/a
S/B = 1.00	n/a n/a	n/a n/a	n/a 0	n/a n/a
S/B = 1.03	n/a n/a	n/a n/a	n/a 0	n/a n/a
S/B = 1.125	n/a n/a	n/a n/a	n/a 0	n/a n/a
γ with minimum MAPE	Maturity (year)			
	1/12	1/4	1/2	1
S/X = 0.90	n/a n/a	n/a n/a	n/a -0.1	n/a 0.1
S/X = 0.97	-5.0 -0.1	-5.0 0.2	-5.0 0.2	-5.0 0.1
S/X = 1.00	-5.0 0	-5.0 0.5	-5.0 0.2	-5.0 0.1
S/X = 1.03	-5.0 0.5	-5.0 0.5	-5.0 0.2	-5.0 0.1
S/X = 1.125	-5.0 0.5	-5.0 0.5	-5.0 0.2	-5.0 0.1

Table 5: Optimal Cressie-Read discrepancy function for SVCJ model.

This table contains the Cressie-Read discrepancy function which attains the minimum error for the SVCJ model. Each cell is associated with an European Option Call with a different combination of moneyness and maturity. The first panel displays the γ in which the method has zero mean percentage error (MPE). The second panel displays the γ in which the method has the lowest mean absolute percentage error (MAPE). The index γ defines the Cressie-Read function through the function $CR_\gamma(\cdot)$ - see equation (4). Appendix B shows the graphs from where those values are obtained. Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The entries with n/a means that no γ in the range $(-5, 2)$ has zero MPE.

γ with MPE equal to zero	Maturity (year)			
	1/12	1/4	1/2	1
S/B = 0.90	n/a	n/a	-2.3	-1.3
S/B = 0.93	-3.1	-2.2	-1.6	-1.2
S/B = 1.00	-3.1	-2.1	-1.5	-1.1
S/B = 1.03	-3.2	-2.0	-1.5	-1.1
S/B = 1.125	-3.7	-2.0	-1.4	-1.0
γ with minimum MAPE	Maturity (year)			
	1/12	1/4	1/2	1
S/X = 0.90	n/a	n/a	-2.1	-1.3
S/X = 0.93	-1.8	-1.6	-1.5	-1.1
S/X = 1.00	-1.7	-1.5	-1.4	-1.1
S/X = 1.03	-1.6	-1.5	-1.3	-1.0
S/X = 1.125	-1.4	-1.4	-1.2	-0.9

Table 6: Price Interval for $\gamma \in [-4, -0.5]$ for the SVCJ model

This table contains the price interval for a call option in the SVCJ model given by the Cressie Read method when applied with $\gamma \in [-4, -0.5]$. The underlying asset price at t has the fixed value $S = 100$, for all cells but the strike B and maturity h change. Each cell shows the interval $(C_{\gamma=-0.5}, C_{\gamma=-4})$ and the theoretical model-implied option price is given below the interval. Objective and risk-neutral parameters adopted for the SVCJ model appear in Table 1. The value of C_γ is calculated as the average option price obtained using our method when applied to 5000 different realization of the SVCJ process. The theoretical option price is obtained by the average of all 5000 model-implied prices obtained for each realization of the SVCJ model.

	Maturity (year)			
	1/12	1/4	1/2	1
S/B = 0.90	(0.04; 0.04) 0.05	(0.64; 0.75) 0.78	(2.18; 2.67) 2.53	(5.64; 7.18) 6.31
S/B = 0.97	(1.13; 1.19) 1.18	(3.06; 3.39) 3.26	(5.44; 6.25) 5.82	(9.41; 11.33) 10.07
S/B = 1.00	(2.53; 2.63) 2.61	(4.71; 5.10) 4.92	(7.17; 8.08) 7.56	(11.17; 13.18) 11.80
S/B = 1.03	(4.45; 4.57) 4.54	(6.58; 7.02) 6.81	(9.01; 9.99) 9.40	(12.96; 15.02) 13.57
S/B = 1.125	(11.68; 11.75) 11.74	(13.12; 13.55) 13.32	(15.14; 16.13) 15.48	(18.67; 20.69) 19.18

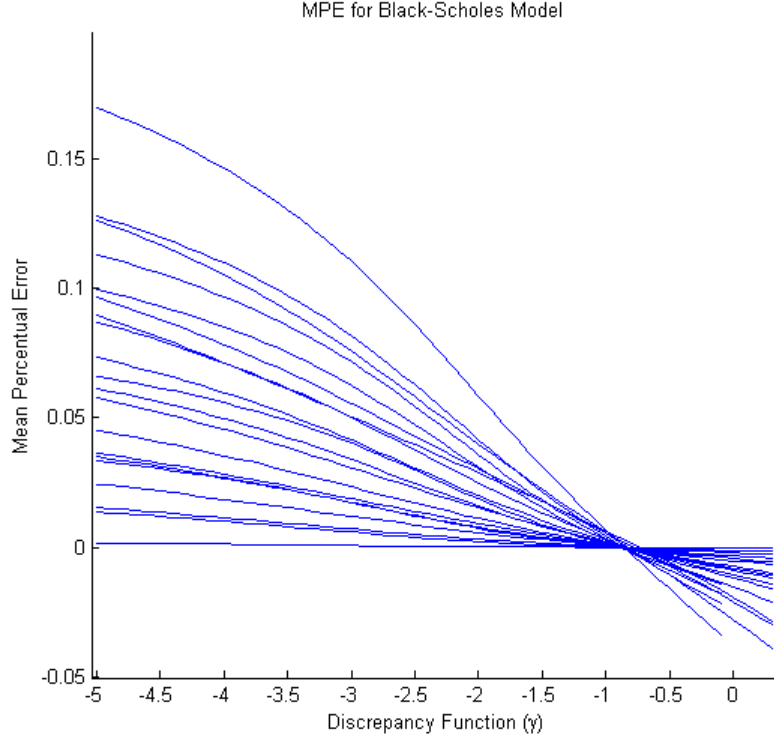


Figure 1: *Graphs of mean percentage errors (MPE) against γ in the Black-Scholes-Merton model. All graphs crosses the horizontal axis close to $\gamma \approx -0.8$. Each curve is associated with one European Call option with a particular combination of maturity and moneyness. Appendix B shows the graphs separately. Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ (equation (4)). Values of interest are: $\gamma \rightarrow -1$ (Empirical Likelihood), $\gamma \rightarrow 0$ (Kullback-Leibler Information Criterion), $\gamma = 1$ (Euclidean estimator).*

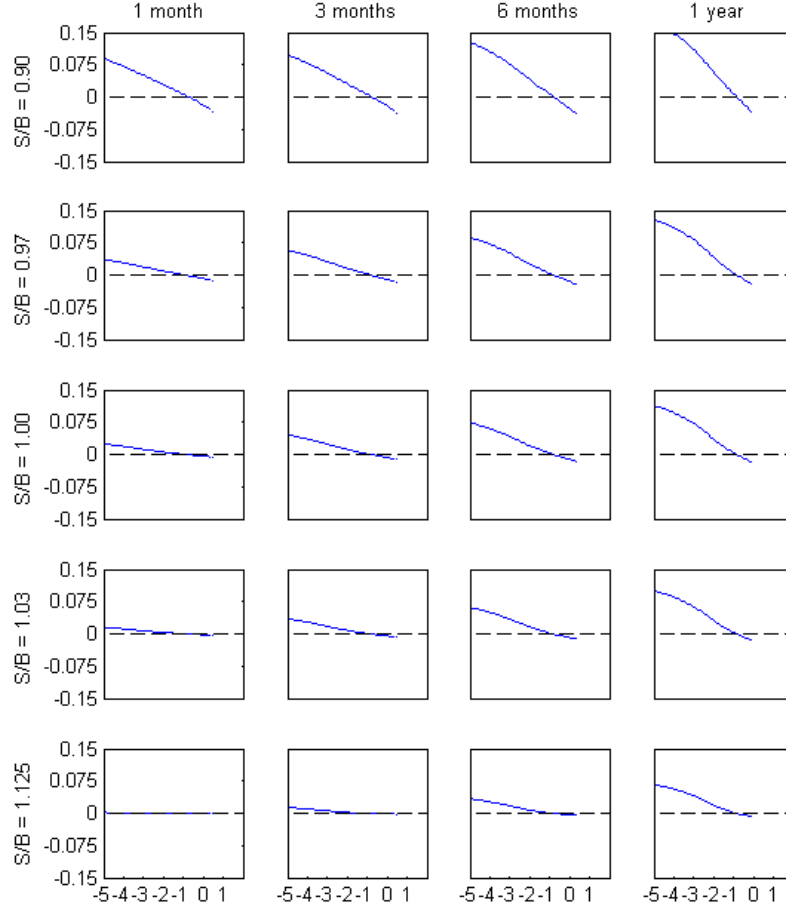


Figure 2: Graphs of mean percentage errors (MPE) against γ in the Black-Scholes-Merton model. All graphs crosses the horizontal axis close to $\gamma \approx -0.8$. Figure 1 is obtained superimposing all those cells. Each cell corresponds to the pricing error of a European Call option with a particular combination of maturity and moneyness (S/B). The parameter for B&S model are $\mu = 10\%$, $\sigma = 20\%$ and $r = 5\%$. Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ - see equation (4). We use only one Euler equation in the restrictions (equation (9)) and don't take into account any derivative price.

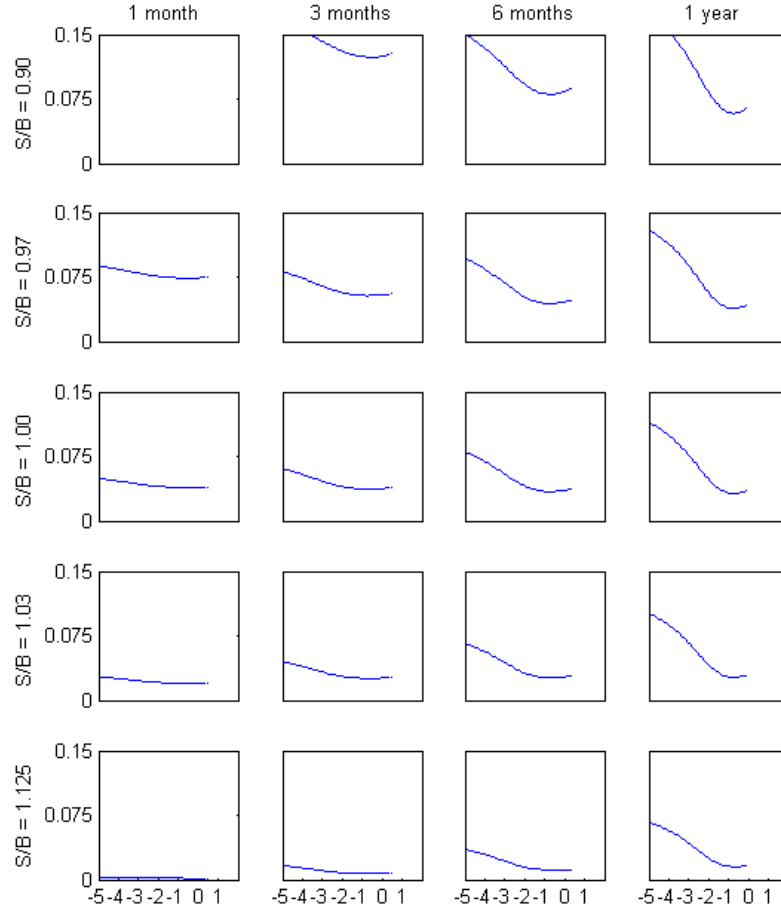


Figure 3: *Graphs of mean absolute percentage errors (MAPE) against γ in the Black-Scholes-Merton model. Each cell corresponds to the pricing error of a European Call option with a particular combination of maturity and moneyness (S/B). The parameter for B&S model are $\mu = 10\%$, $\sigma = 20\%$ and $r = 5\%$. Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ - see equation (4). We use only one Euler equation in the restrictions (equation (9)) and don't take into account any derivative price.*

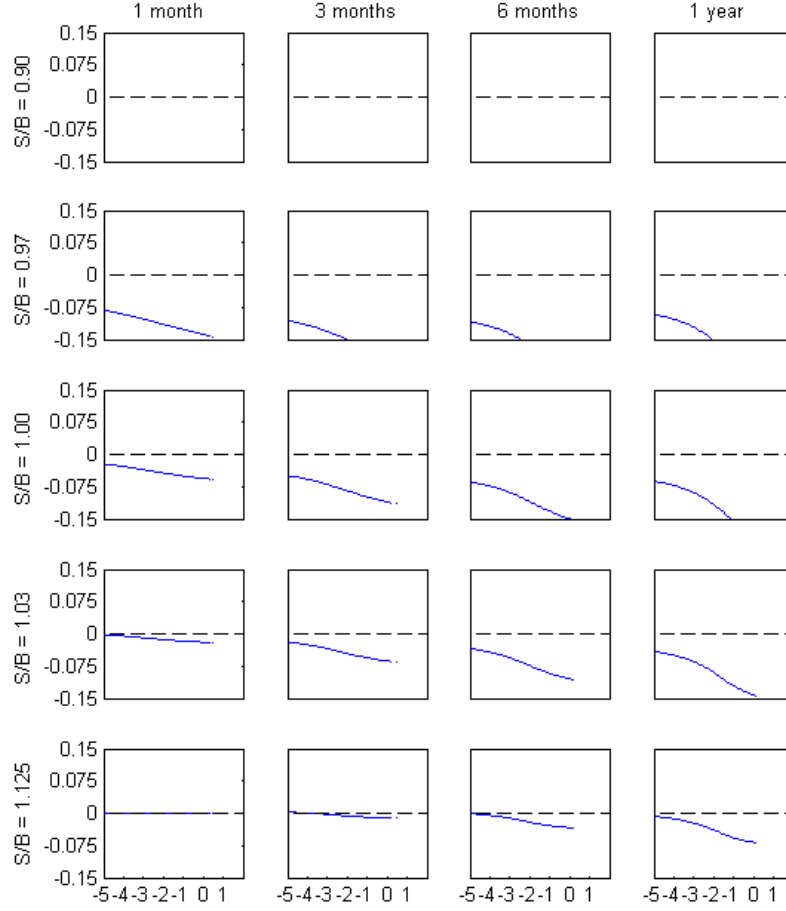


Figure 4: Graphs of mean percentage errors (MPE) against γ in the Stochastic Volatility (Heston) model with volatility risk premium. The parameters are as estimated in Garcia, Lewis, Pastorello and Renaults (2011) and may be found in the Table 2 for the reader's convenience. Each cell corresponds to the pricing error of a European Call option with a particular combination of maturity and moneyness (S/B). Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ - see equation (4). We use only one Euler equation in the restrictions (equation (9)) and don't take into account any derivative price.

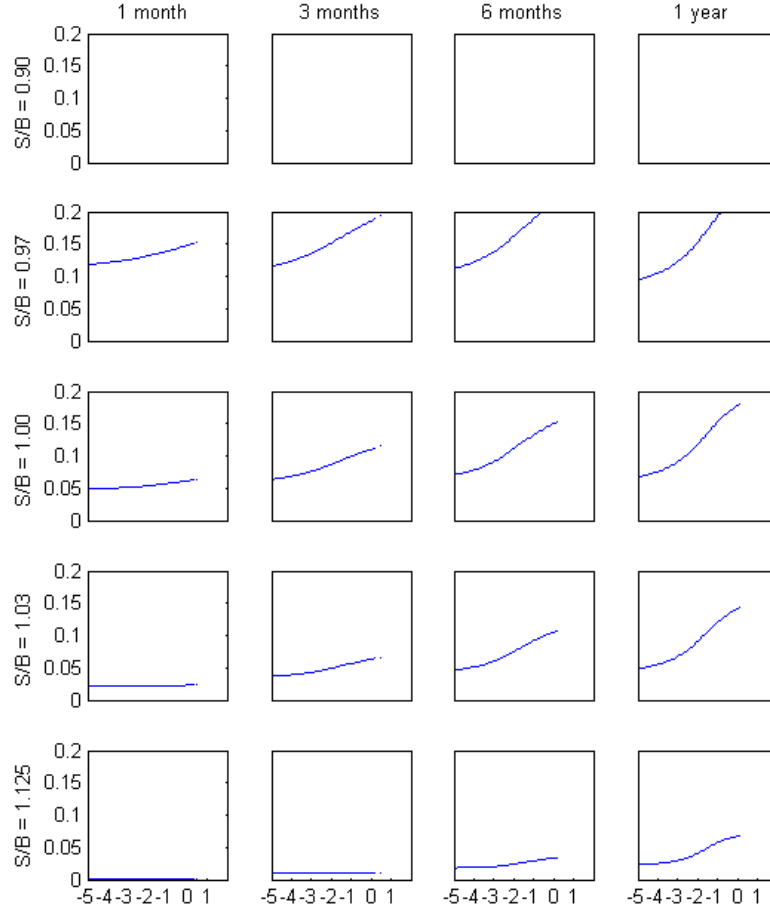


Figure 5: Graphs of mean absolute percentage errors (MAPE) against γ in the Stochastic Volatility (Heston) model with volatility risk premium. The parameters are as estimated in Garcia, Lewis, Pastorello and Renaults (2011) and may be found in the table 2 for the reader's convenience. Each cell corresponds to the pricing error of a European Call option with a particular combination of maturity and moneyness (S/B). Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ - see equation (4). We use only one Euler equation in the restrictions (equation (9)) and don't take into account any derivative price.

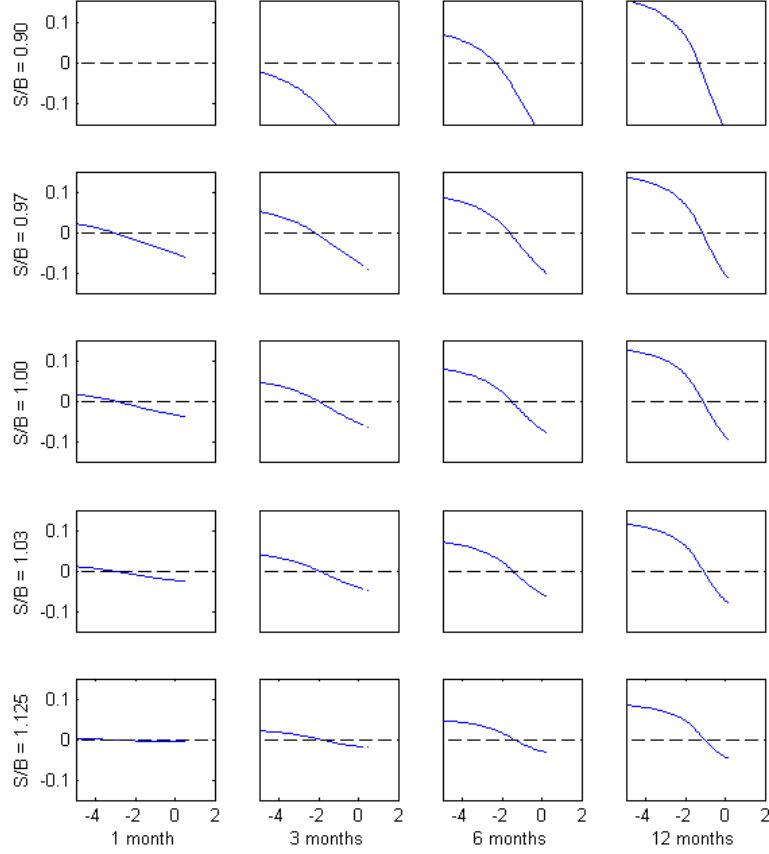


Figure 6: *Graphs of mean percentage errors (MPE) against γ in the SVCJ model. Each cell corresponds to the pricing error of a European Call option with a particular combination of maturity and moneyness (S/B). Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ - see equation (4). We use only one Euler equation in the restrictions (equation (9)) and don't take into account any derivative price.*

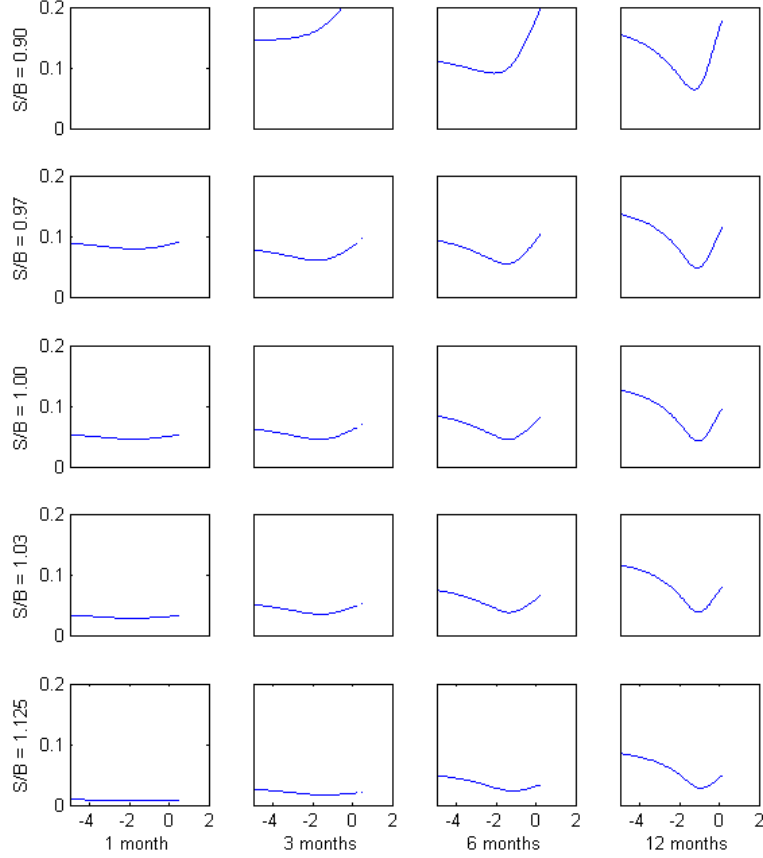


Figure 7: *Graphs of mean absolute percentage errors (MAPE) against γ in the SVCJ model. Each cell corresponds to the pricing error of a European Call option with a particular combination of maturity and moneyness (S/B). Pricing errors are based on the method applied to 200 returns draws from appropriate distribution and the mean is obtained by an average of 5000 repetitions. The parameter γ defines the discrepancy function through the function $CR_\gamma(\cdot)$ - see equation (4). We use only one Euler equation in the restrictions (equation (9)) and don't take into account any derivative price.*