Machine Learning Methods in Empirical Finance

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Lecture 2
XVIII Encontro Brasileiro de Finanças

Regression Trees

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Introduction

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- Recursive partition of the input space $\mathbb{X}$ (set of explanatory variables).

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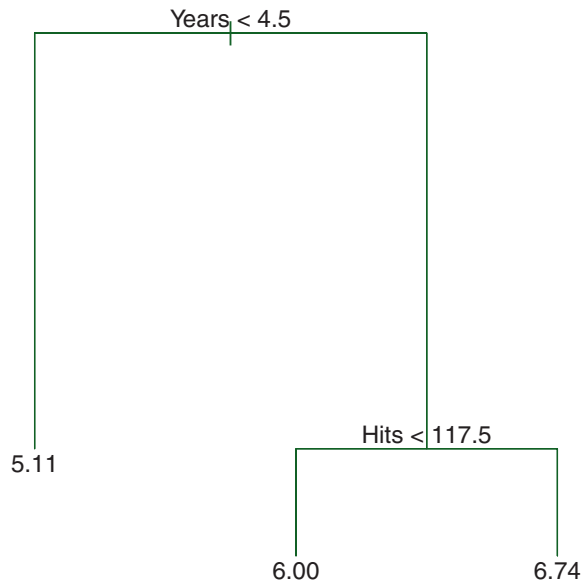
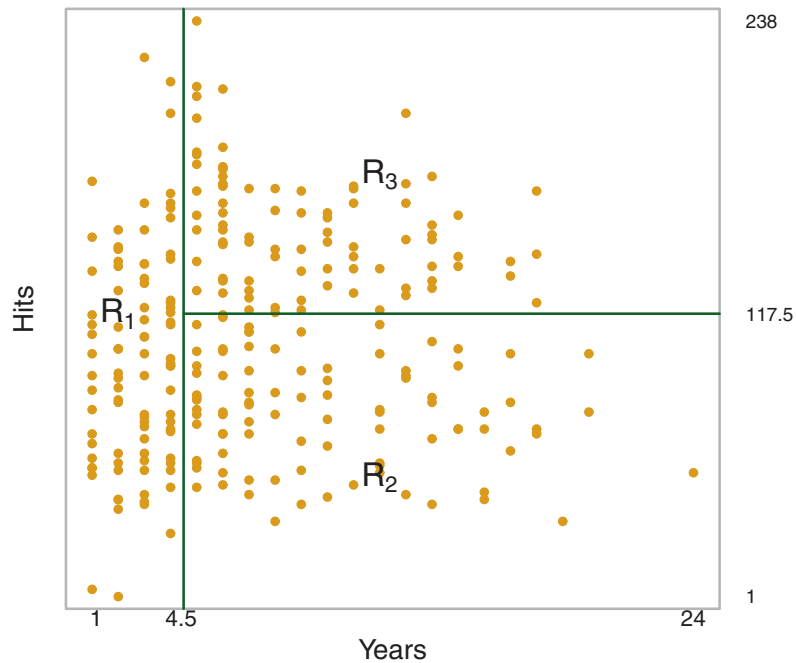
- Hierarchical nature
- Interpretability
- Categorical variables and missing data (treated as a category) easily handled.
- Main disadvantage: **highly unstable**.
  - Easy fixes: *bagging* and *boosting*
  - Drawback: interpretability lost.

Regression Trees

Example

- Predict log wages of baseball players based on the length of the career (*years*) and number of *hits*.





Regression Trees

Mathematical Representation

Notation:

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- $\mathbb{J}$ and $\mathbb{T}$ are the sets of parent and terminal nodes, respectively.

Regression Trees

Mathematical Representation

$$y_t = H_{\mathbb{J}\mathbb{T}}(\boldsymbol{x}_t; \boldsymbol{\psi}) + u_t = \sum_{i=1}^{K} \beta_i I(\boldsymbol{x}_t \in \mathcal{R}_i) + u_t = \sum_{i \in \mathbb{T}} \beta_i B_{\mathbb{J}i}\left(\boldsymbol{x}_t; \boldsymbol{\theta}_i\right) + u_t$$

$$B_{\mathbb{J}i}\left(\boldsymbol{x}_t; \boldsymbol{\theta}_i\right) = \prod_{j \in \mathbb{J}} I(x_{s_j,t}; c_j)^{\frac{n_{i,j}(1+n_{i,j})}{2}} \left[1 - I(x_{s_j,t}; c_j)\right]^{(1-n_{i,j})(1+n_{i,j})},$$

$$I(x_{s_j,t}; c_j) = \begin{cases} 1 & \text{if } x_{s_j,t} \leq c_j \\ 0 & \text{otherwise,} \end{cases}$$

$$n_{i,j} = \begin{cases} -1 & \text{if the path to leaf } i \text{ does not include parent node } j; \\ 0 & \text{if the path to leaf } i \text{ include the right-hand child of parent node } j; \\ 1 & \text{if the path to leaf } i \text{ include the left-hand child of parent node } j. \end{cases}$$

- $\mathbb{J}_i$: set of indexes of parent nodes included in the path to leaf i.
- $\boldsymbol{\theta}_i = \{c_k\}$ such that $k \in \mathbb{J}_i$, $i \in \mathbb{T}$ e $\sum_{j \in \mathbb{J}} B_{\mathbb{J}i}\left(\boldsymbol{x}_t; \boldsymbol{\theta}_j\right) = 1$.

Regression Trees

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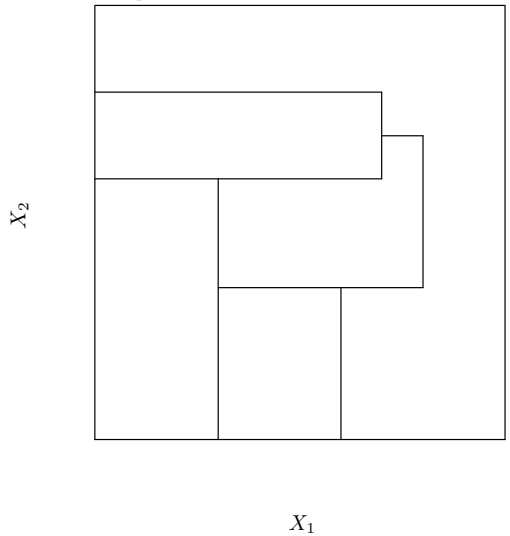
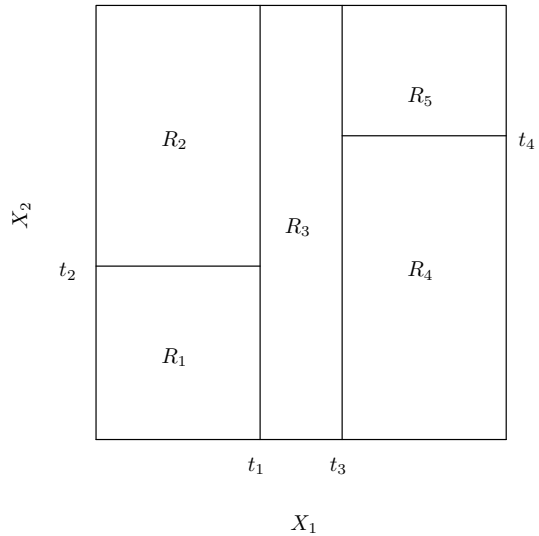
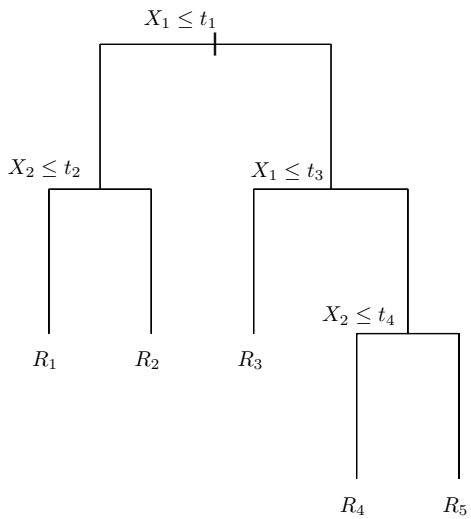
- Recursive partitions are created until the total sum of squares is bellow a certain pre-specified value or the number of observations in each region reached a minimum value.

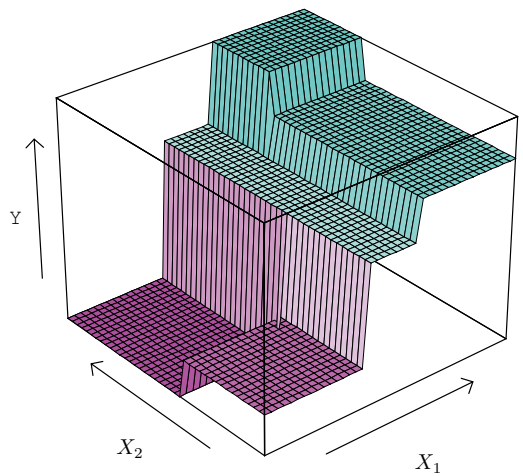
Regression Trees

Recursive Partitioning: Example









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$$C_\alpha(\mathcal{T}) = \text{sum of squared errors} + \alpha K,$$

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- α is selected by cross-validation.

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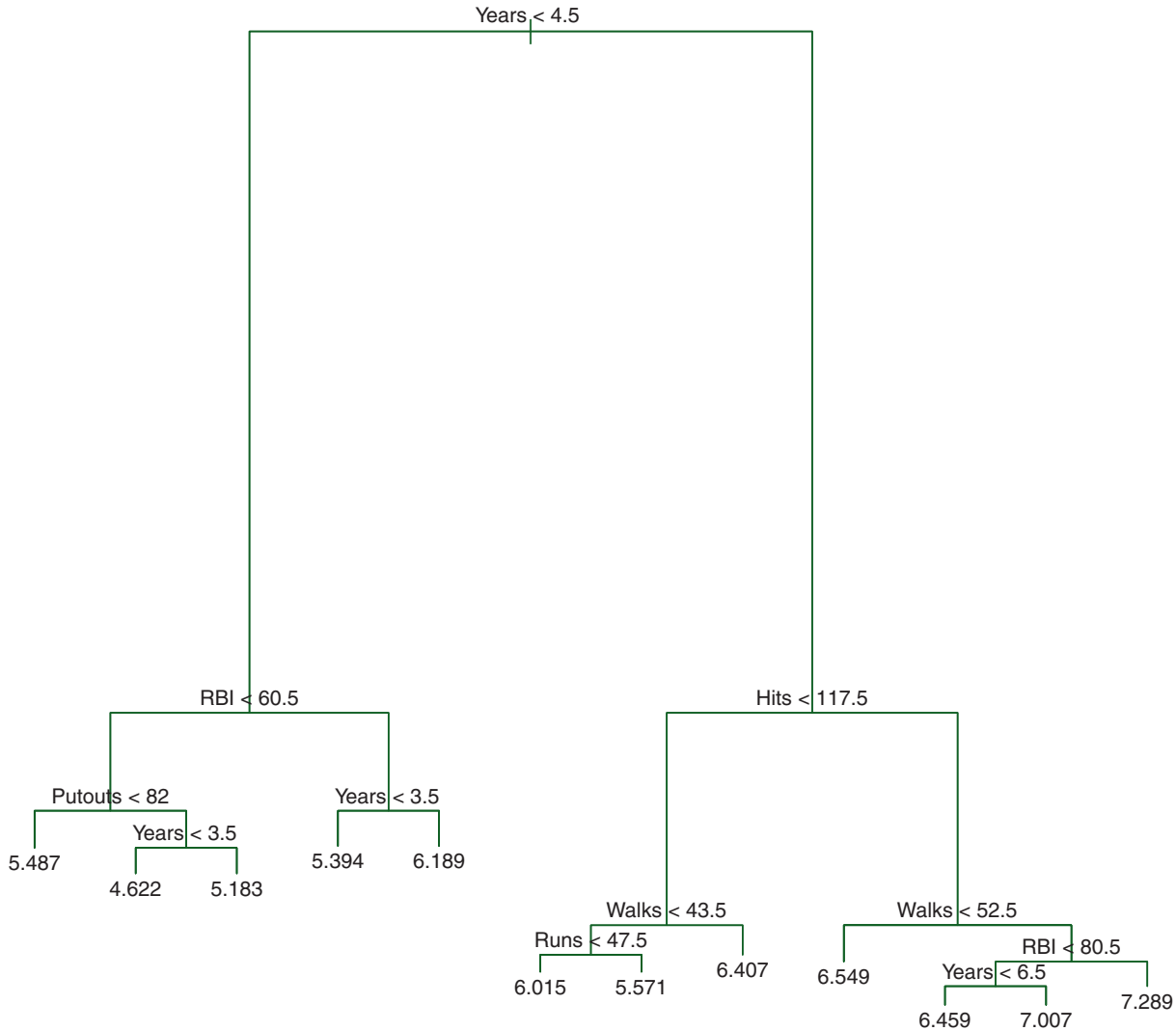

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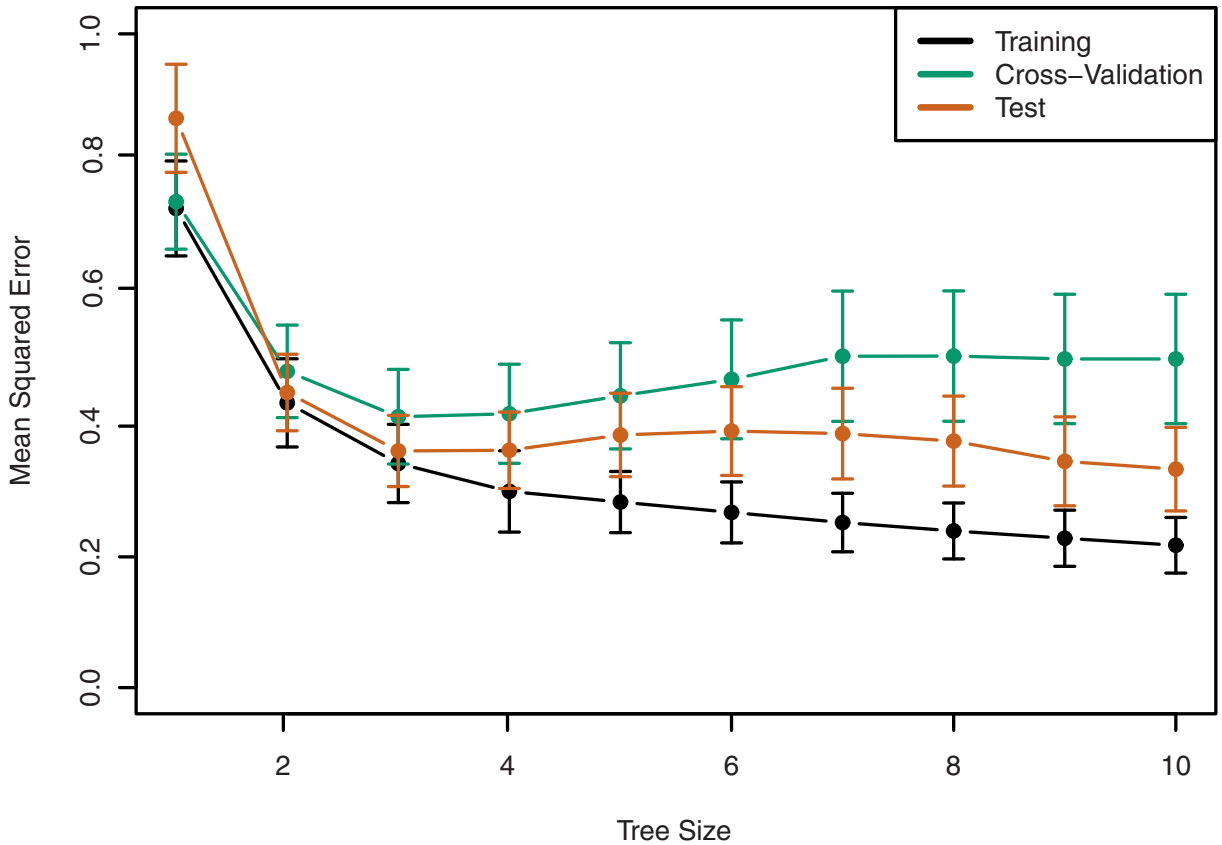

Boosting Regression Trees

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  - Additive models:

$$y_t = f(\boldsymbol{x}_t) + u_t = \sum_{m=1}^{M} \beta_m f_m(\boldsymbol{x}_t) + u_t,$$

where:

- y_t: dependent variable;
- $\boldsymbol{x}_t \in \mathbb{R}^p$: vector of explanatory variables;
- u_t: random error;
- $f_m(\boldsymbol{x}_t)$: basis function or *weak learner*.

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- Origin of the method: classification problems.

Introduction to Boosting

- The goal is to estimate the optimal function f^*, defined as

$$\boxed{f^* := \arg\min_f \mathbb{E}\left[L(\boldsymbol{Y}, f(\boldsymbol{X}))\right],}$$

where:

- $\boldsymbol{Y} = (y_1, \ldots, y_t)'$
- $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p)$ and $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{iT})'$ and
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  - $L(\cdot, \cdot)$ is a **loss function**.
- We consider just the **quadratic loss**: L_2-Boosting.
- Therefore, the goal is to estimate $\mathbb{E}(\boldsymbol{Y}|\boldsymbol{X})$.

Introduction to Boosting

- In practice, the optimization problem has the following form:

$$\widehat{f} = \arg\min_{f} \underbrace{\frac{1}{T}\sum_{t=1}^{T} L(y_t, f(\boldsymbol{x}_t))}_{:=\mathcal{R}},$$

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- For quadratic loss:

$$\mathcal{R} = \frac{1}{T}\sum_{t=1}^{T} \left[y_t - f(\boldsymbol{x}_t)\right]^2.$$

Properties

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- Robust to multi-collinearity.
- Optimizes the predictive power.

Gradient Boosting

Algorithm:

1. Initialize a T-dimensional vector $\widehat{\boldsymbol{f}}^{[0]}$ with some value. Set $j = 0$ and choose the weak learners. Set the number of learners to p.
2. Set $j = j + 1$ and compute

$$-\frac{\partial}{\partial \boldsymbol{f}} L(\boldsymbol{Y}, \boldsymbol{f}) \quad \text{and} \quad \widehat{\boldsymbol{f}}^{[j-1]}(\boldsymbol{x}_t),\ t = 1, \ldots, T.$$

Write

$$\begin{aligned} \boldsymbol{U}^{[j-1]} &= \left[U_t^{[j-1]}\right]_{t=1,\ldots,T} \\ &:= \left[-\frac{\partial}{\partial \boldsymbol{f}} L(\boldsymbol{Y}, \boldsymbol{f}) \Big|_{\boldsymbol{Y}=y_t, \boldsymbol{f}=\widehat{f}^{[j-1]}(\boldsymbol{x}_t)} \right]_{t=1,\ldots,T} \end{aligned}$$

Gradient Boosting

Algorithm (continuation):

3. Compute $\boldsymbol{U}^{[j-1]}$ and choose the learner that best fits $\boldsymbol{U}^{[j-1]}$. Set $\widehat{\boldsymbol{U}}^{[j-1]}$ as the fitted values of the best model.
4. Update

$$\widehat{\boldsymbol{f}}^{[j]} = \widehat{\boldsymbol{f}}^{[j-1]} + \nu \widehat{\boldsymbol{U}}^{[j-1]}, \quad 0 < \nu \leq 1.$$

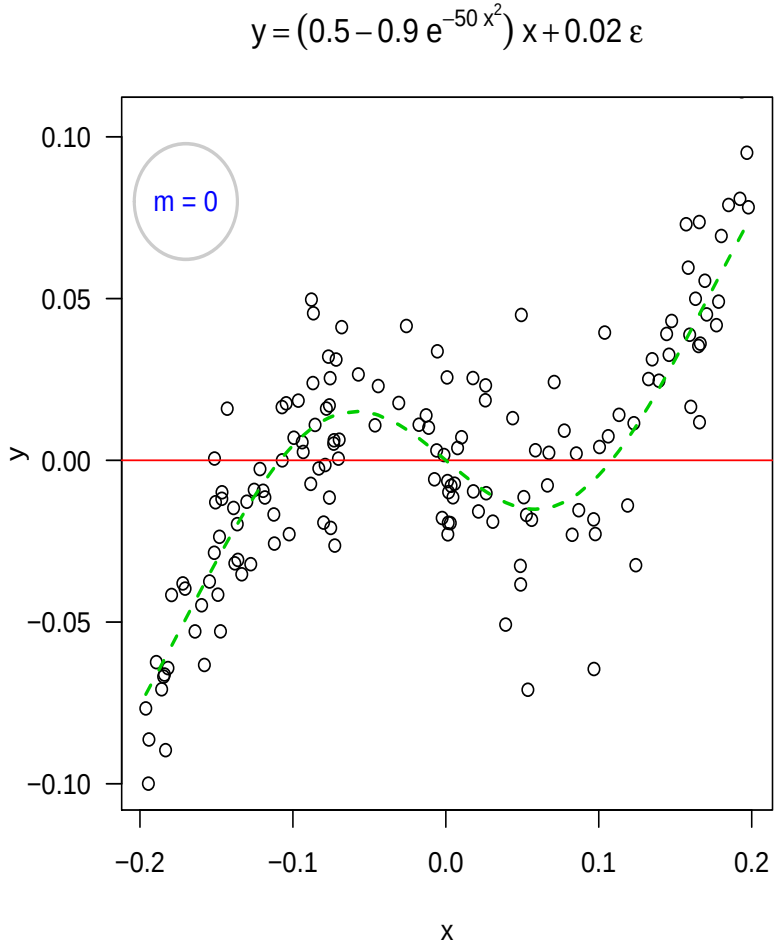
5. Repeat steps 2-4 until the maximum number of iterations is reached.

Componentwise Algorithm

- One learner for each covariate.

$$\begin{aligned}
\boldsymbol{U}^{[j-1]} &\sim \boldsymbol{x}_1 \\
\boldsymbol{U}^{[j-1]} &\sim \boldsymbol{x}_2 \\
\boldsymbol{U}^{[j-1]} &\sim \boldsymbol{x}_3 \\
&\vdots \\
\boldsymbol{U}^{[j-1]} &\sim \boldsymbol{x}_k \longrightarrow \text{Best Model} \longrightarrow \widehat{\boldsymbol{U}}^{[j-1]} \\
&\vdots \\
\boldsymbol{U}^{[j-1]} &\sim \boldsymbol{x}_p
\end{aligned}$$

Example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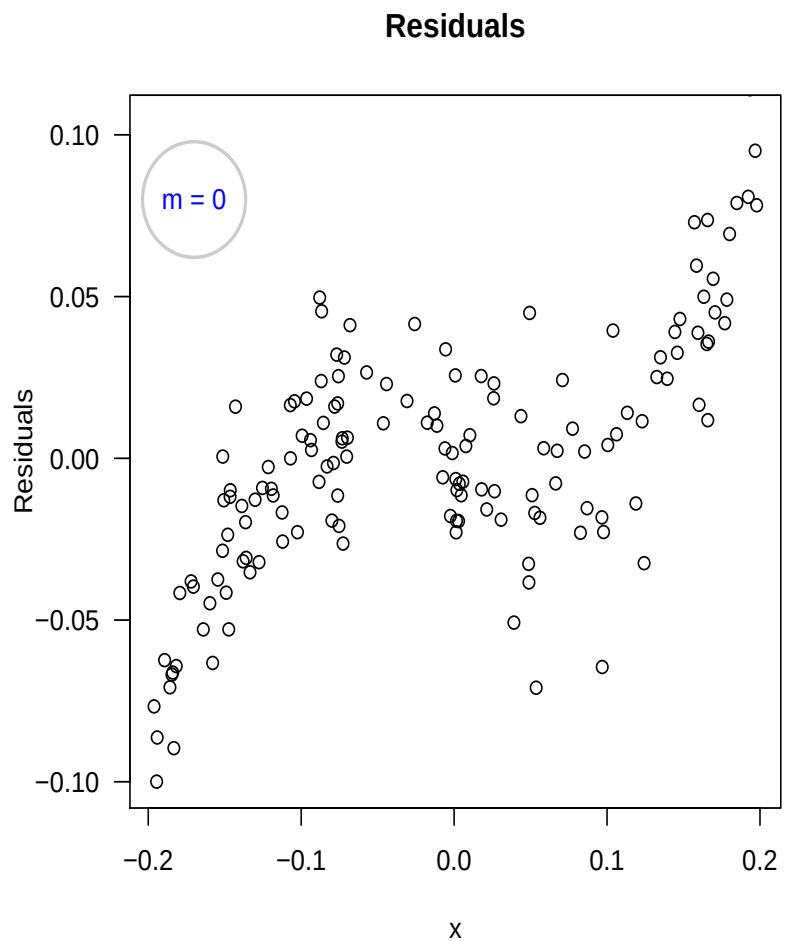
m = 0
0.10
0.05
0.00
−0.05
−0.10
y
−0.2
−0.1
0.0
0.1
0.2
x


Residuals
m = 0
0.10
0.05
0.00
−0.05
−0.10
Residuals
−0.2
−0.1
0.0
0.1
0.2
x

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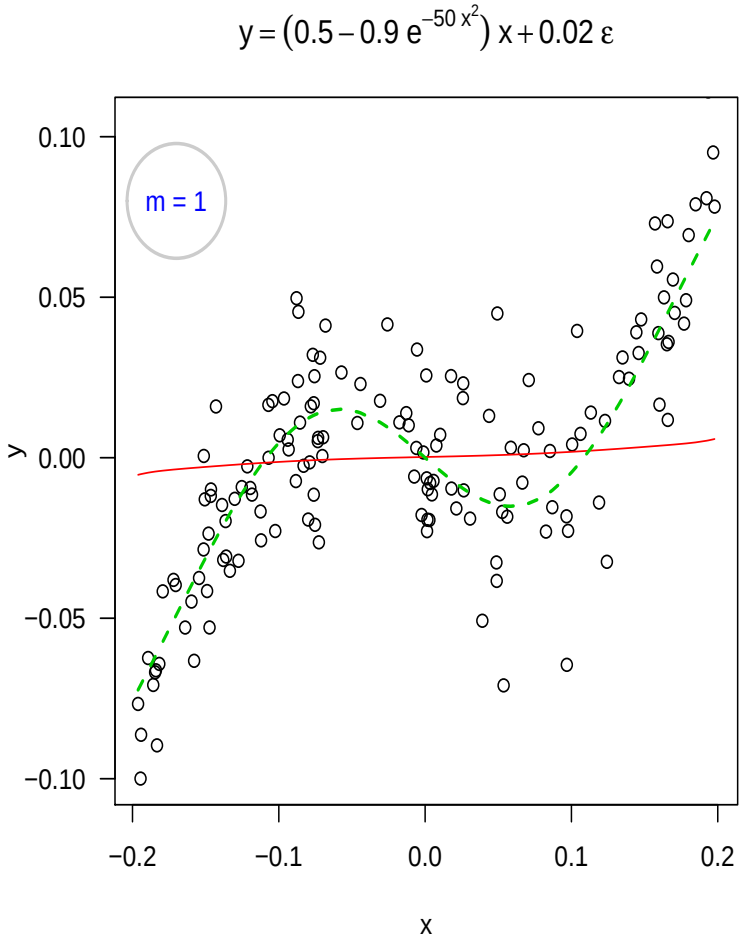
$y = (0.5 - 0.9\, e^{-50\, x^2})\, x + 0.02\, \varepsilon$
m = 1
0.10
0.05
0.00
−0.05
−0.10
y
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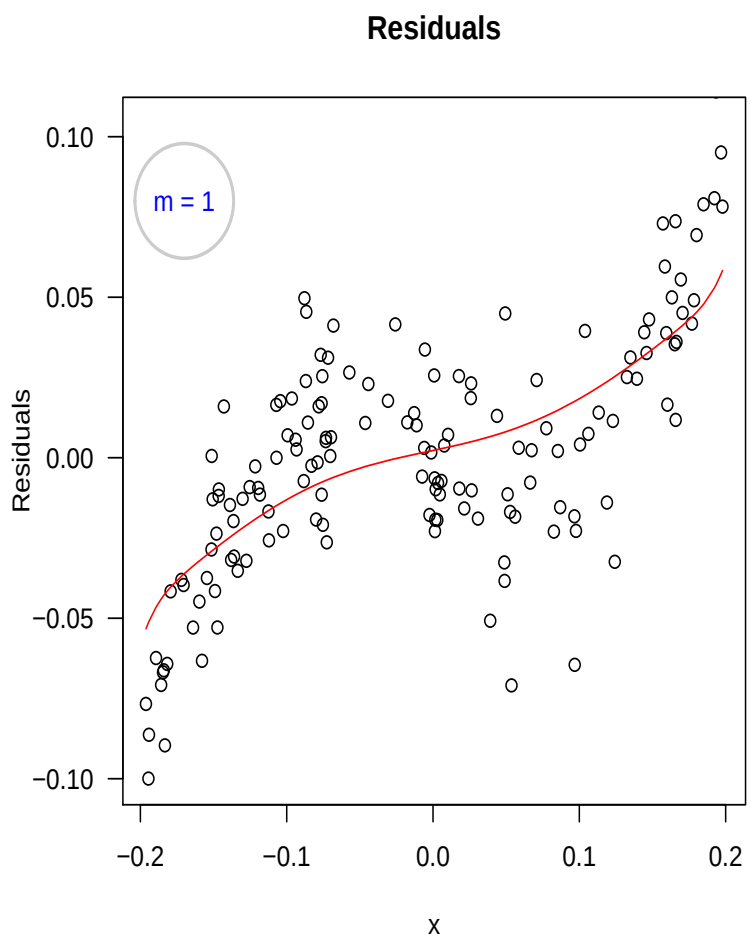
Residuals
m = 1
0.10
0.05
0.00
−0.05
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Residuals
−0.2
−0.1
0.0
0.1
0.2
x

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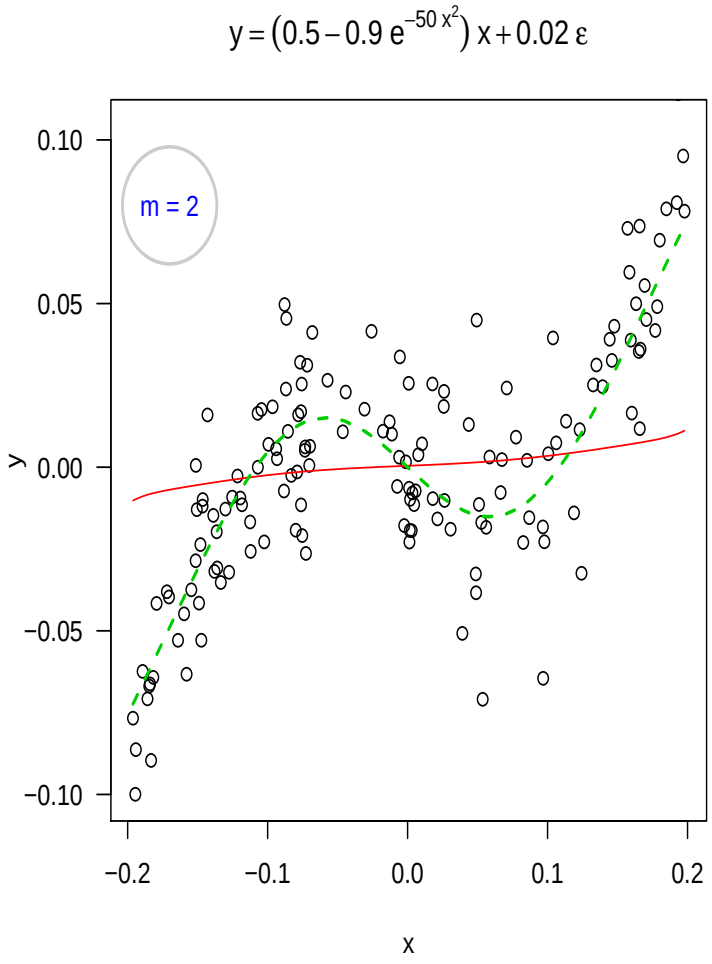
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m = 2
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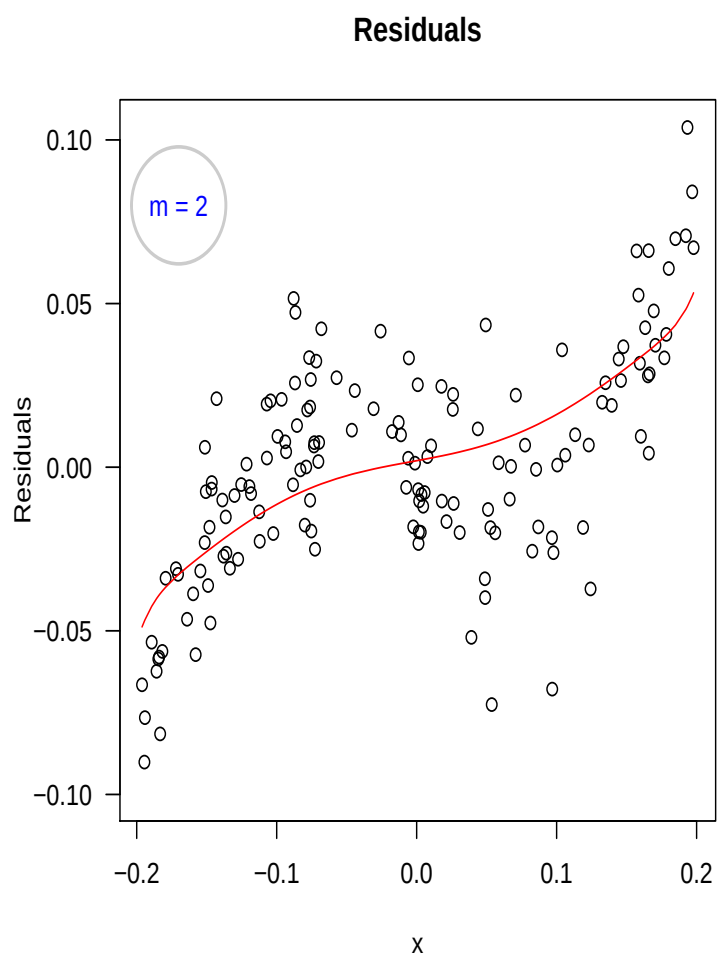
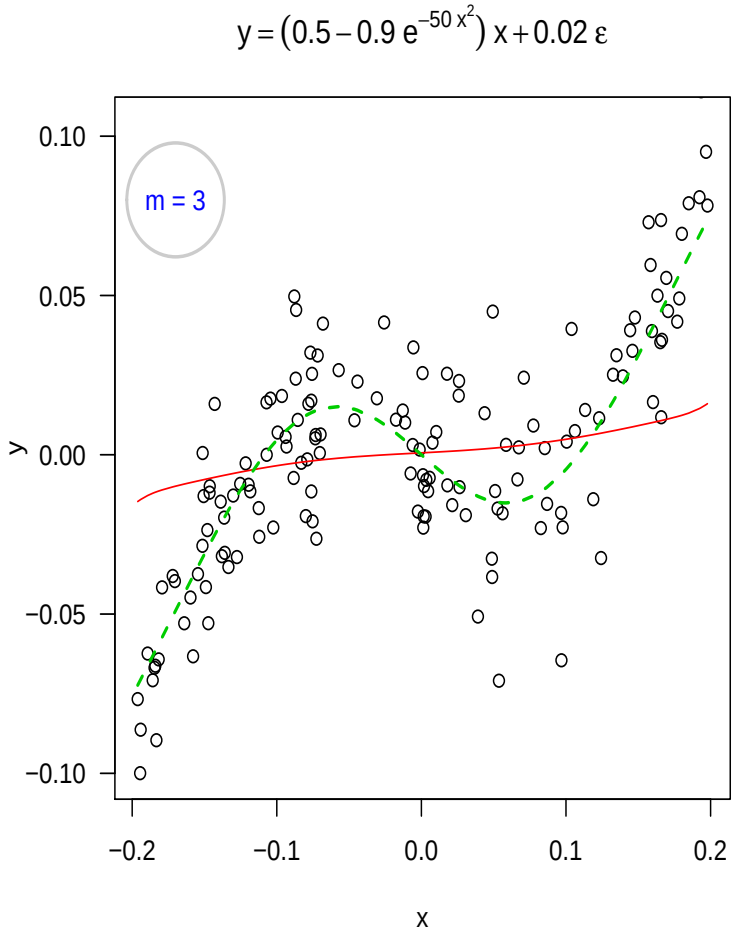
0.05
0.00
−0.05
−0.10
y
−0.2
−0.1
0.0
0.1
0.2
x


Residuals
m = 2
0.10
0.05
0.00
−0.05
−0.10
Residuals
−0.2
−0.1
0.0
0.1
0.2
x

Example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m = 3
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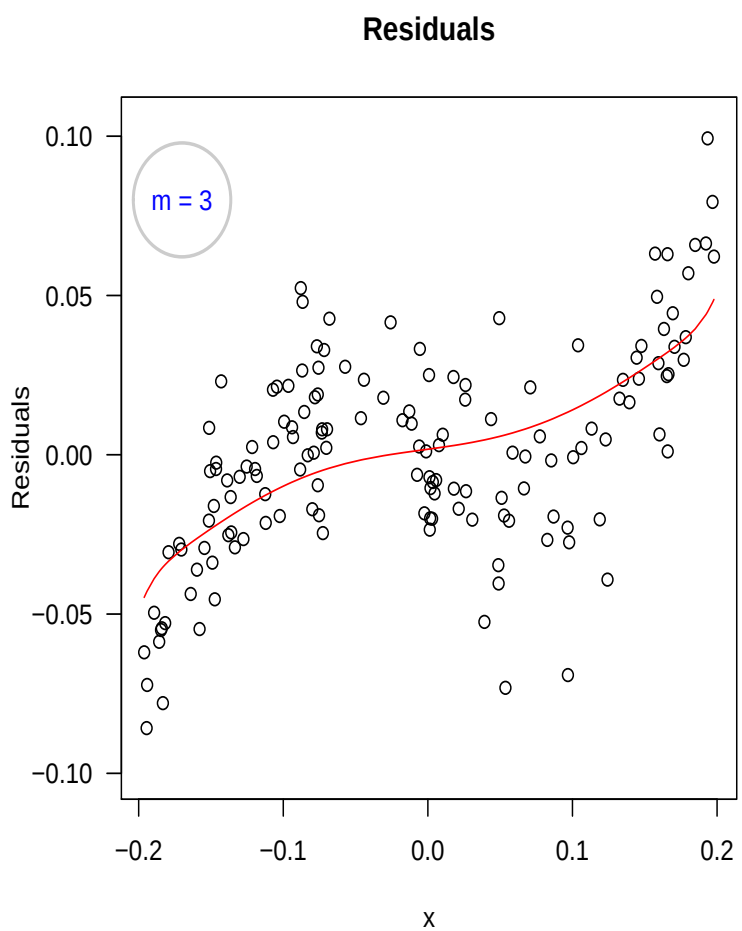
0.00
−0.05
−0.10
y
−0.2
−0.1
0.0
0.1
0.2
x


Residuals
m = 3
0.10
0.05
0.00
−0.05
−0.10
Residuals
−0.2
−0.1
0.0
0.1
0.2
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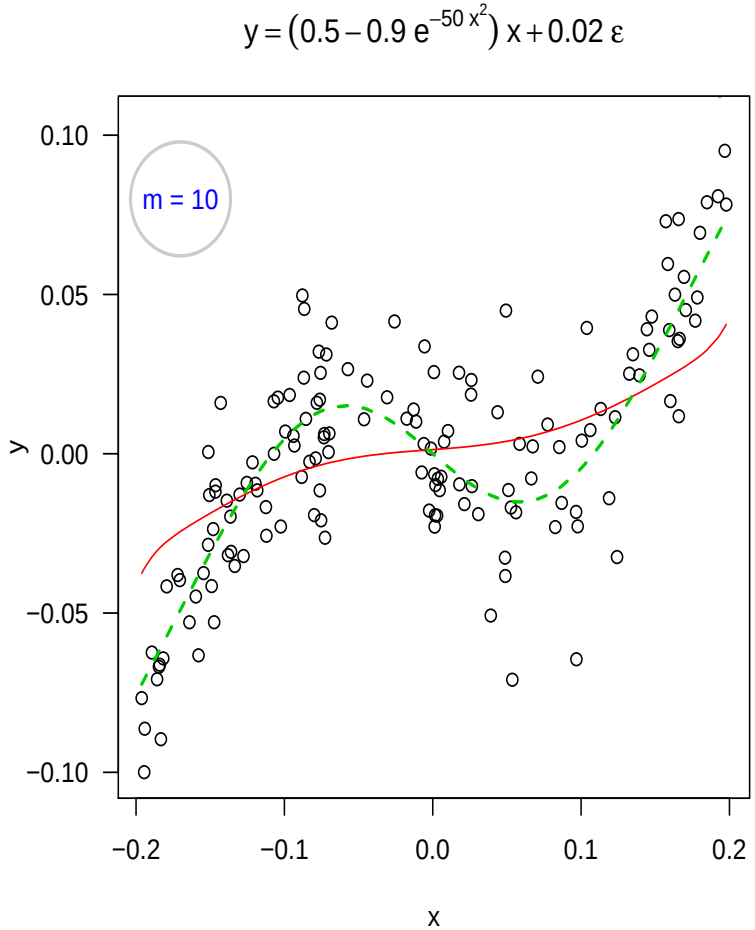
$y = (0.5 - 0.9\, e^{-50\, x^2})\, x + 0.02\, \varepsilon$
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−0.10
y
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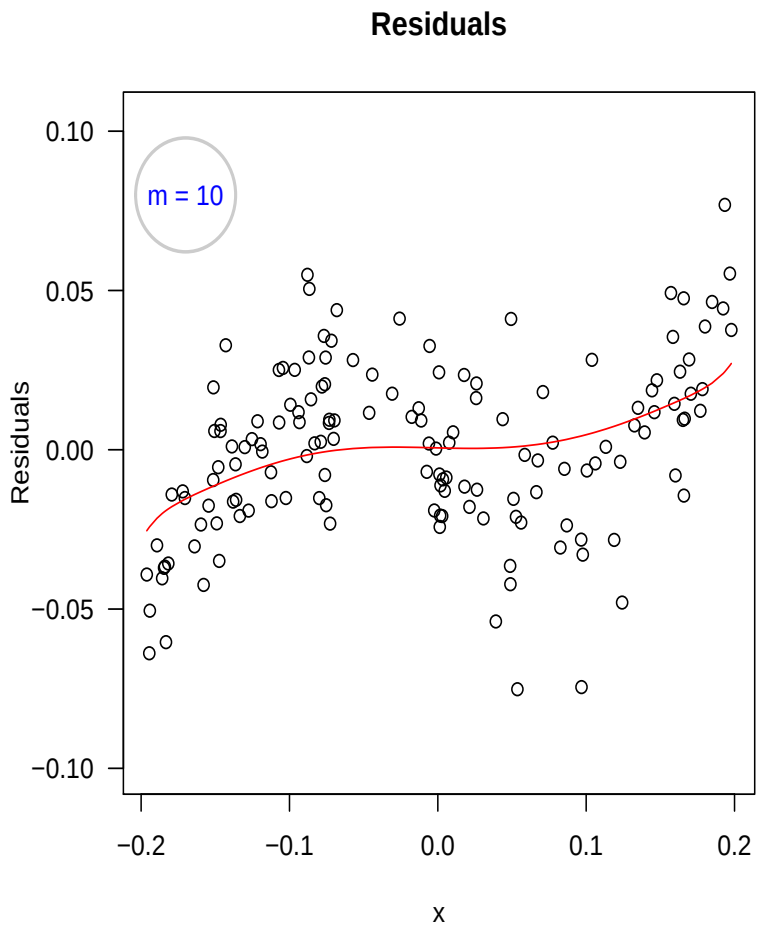
−0.1
0.0
0.1
0.2
x


Residuals
m = 10
0.10
0.05
0.00
−0.05
−0.10
Residuals
−0.2
−0.1
0.0
0.1
0.2
x

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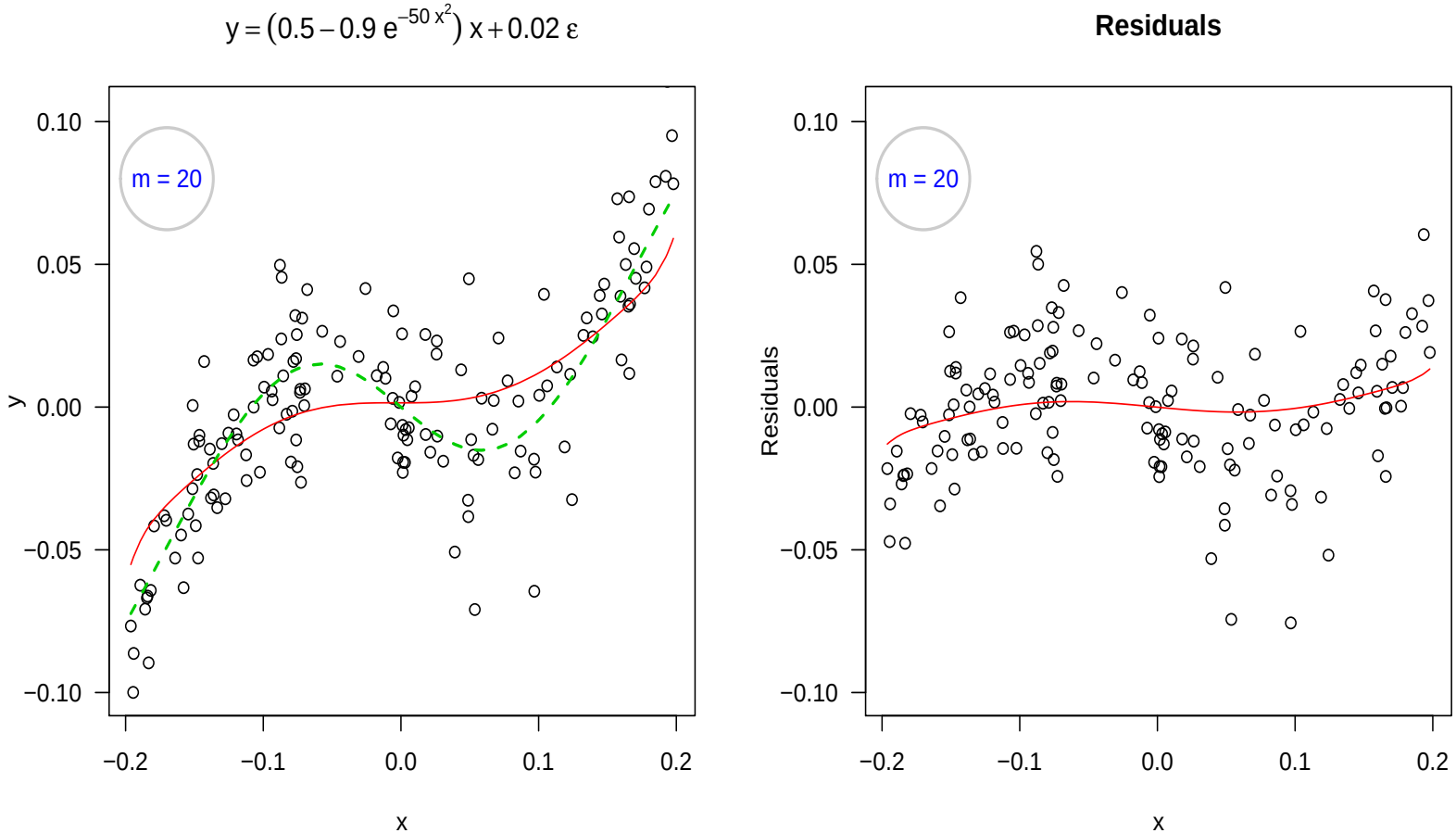
$y = (0.5 - 0.9\, e^{-50\, x^2})\, x + 0.02\, \varepsilon$
m = 20
y
x
Residuals
m = 20
Residuals
x

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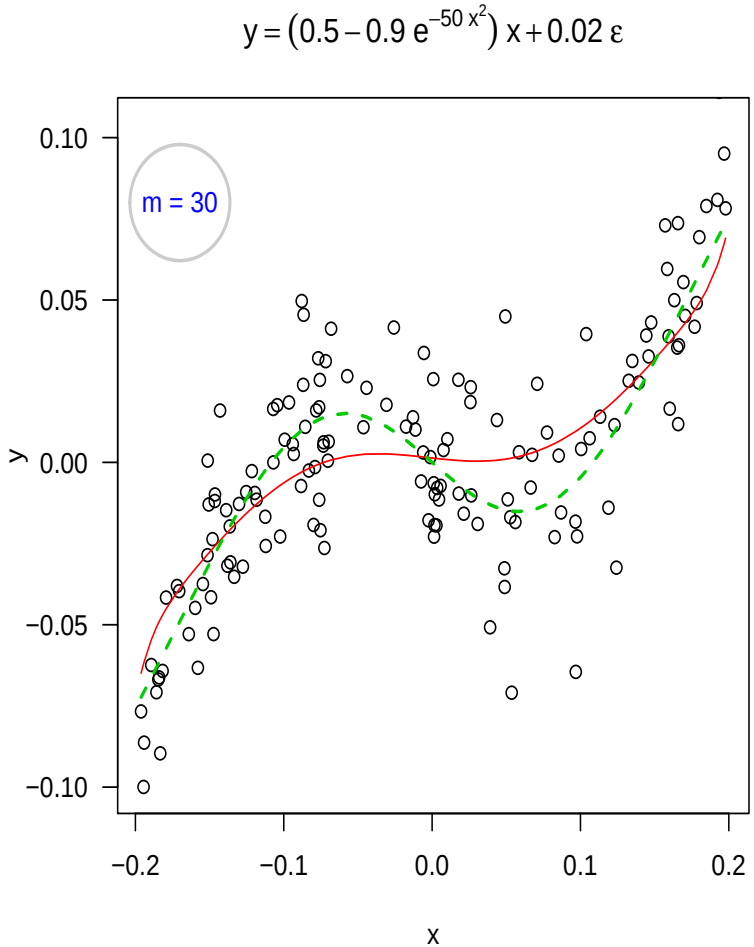
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0.10
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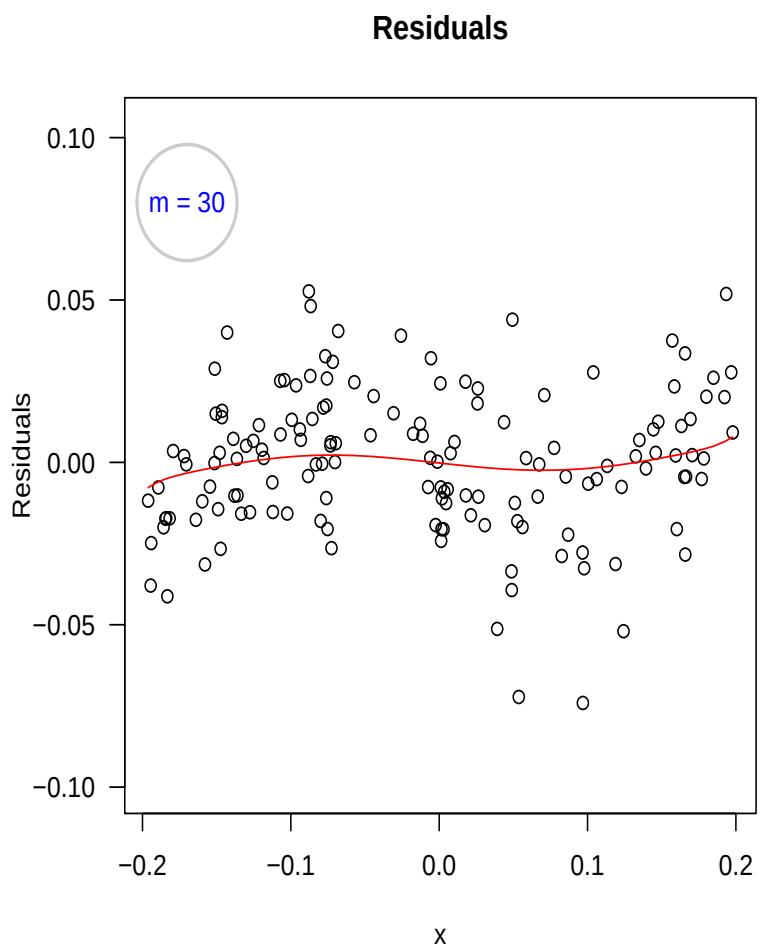
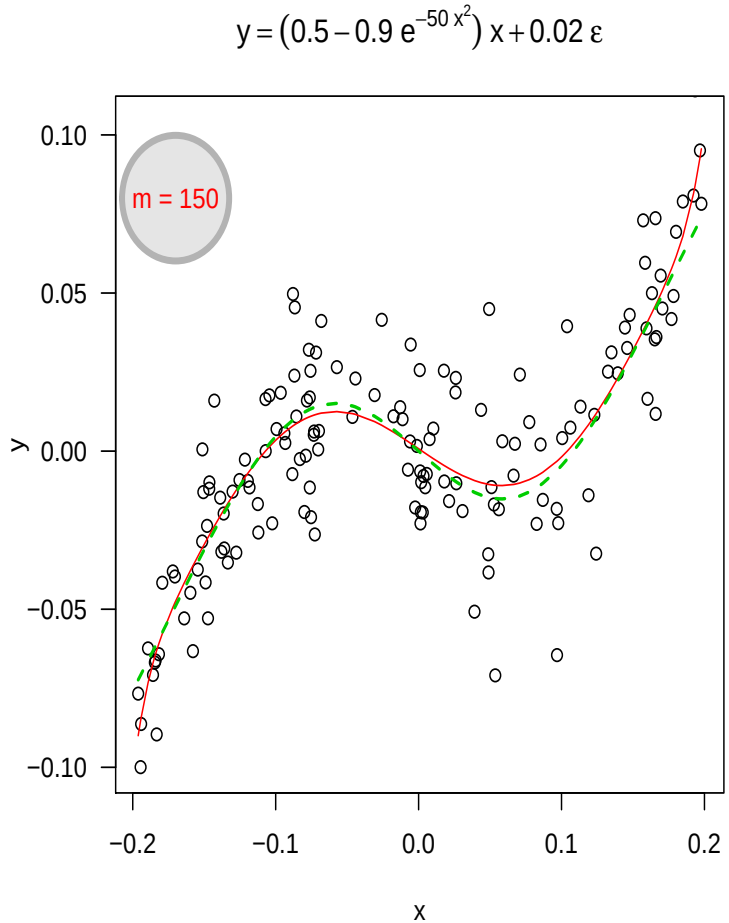
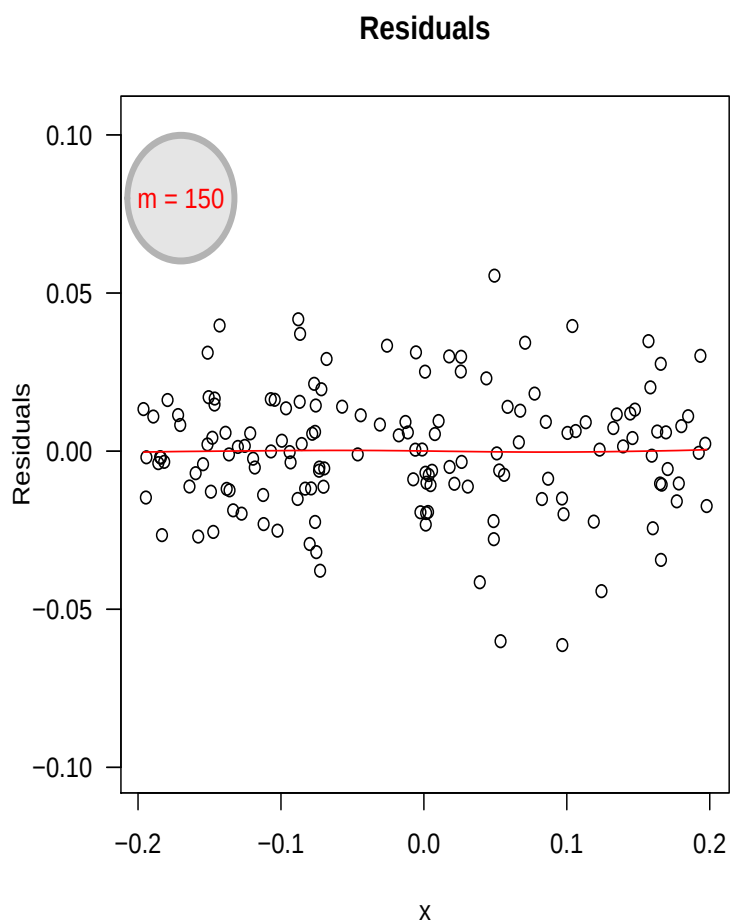
x


Residuals
m = 30
0.10
0.05
0.00
−0.05
−0.10
Residuals
−0.2
−0.1
0.0
0.1
0.2
x

Example


$y = (0.5 - 0.9\, e^{-50\, x^2})\, x + 0.02\, \varepsilon$
m = 150
0.10
0.05
0.00
−0.05
−0.10
y
−0.2
−0.1
0.0
0.1
0.2
x


Residuals
m = 150
0.10
0.05
0.00
−0.05
−0.10
Residuals
−0.2
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0.0
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x

Gradient Boosting Properties

- It is clear from step 4 that

$$\boxed{\widehat{\boldsymbol{f}}^{[J]} = \widehat{\boldsymbol{f}}^{[0]} + \nu\widehat{\boldsymbol{U}}^{[0]} + \cdots + \nu\widehat{\boldsymbol{U}}^{[J-1]}.}$$

- The structure of the prediction function will depend on the choice of the weak learners.
  - Linear learners $\longrightarrow$ linear function
  - Smooth learners $\longrightarrow$ smooth function
- Variable selection
- Useful when $p >> T$.
- If $J \longrightarrow \infty$, the algorithm will select irrelevant regressors.
  - Solution: *early stopping* with cross-validation or information criteria.

Example: Linear Learners

Set-up:

- Three regressors: $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$.
- Three linear learners.
- $J = 5$
- Hypothetical solution: $\boldsymbol{x}_1$ selected at iterations 1, 2 and 5 and $\boldsymbol{x}_2$ selected at iterations 3 and 4.

$$
\begin{aligned}
\widehat{\boldsymbol{f}}^{[J]} &= \widehat{\boldsymbol{f}}^{[0]} + \nu\widehat{\boldsymbol{U}}^{[0]} + \nu\widehat{\boldsymbol{U}}^{[1]} + \nu\widehat{\boldsymbol{U}}^{[2]} + \nu\widehat{\boldsymbol{U}}^{[3]} + \nu\widehat{\boldsymbol{U}}^{[4]} \\
&= \widehat{\boldsymbol{\beta}}^{[0]} + \nu\widehat{\boldsymbol{\beta}}_1^{[0]}\boldsymbol{x}_1 + \nu\widehat{\boldsymbol{\beta}}_1^{[1]}\boldsymbol{x}_1 + \nu\widehat{\boldsymbol{\beta}}_3^{[2]}\boldsymbol{x}_3 + \nu\widehat{\boldsymbol{\beta}}_3^{[3]}\boldsymbol{x}_3 + \nu\widehat{\boldsymbol{\beta}}_1^{[4]}\boldsymbol{x}_1 \\
&= \widehat{\boldsymbol{\beta}}^{[0]} + \nu\left(\widehat{\boldsymbol{\beta}}_1^{[0]} + \widehat{\boldsymbol{\beta}}_1^{[1]} + \widehat{\boldsymbol{\beta}}_1^{[4]}\right)\boldsymbol{x}_1 + \nu\left(\widehat{\boldsymbol{\beta}}_3^{[2]} + \widehat{\boldsymbol{\beta}}_3^{[3]}\right)\boldsymbol{x}_3 \\
&= \widehat{\boldsymbol{\beta}}^{[0]} + \widehat{\boldsymbol{\beta}}_1\boldsymbol{x}_1 + \widehat{\boldsymbol{\beta}}_3\boldsymbol{x}_3.
\end{aligned}
$$

Boosting Regression Trees

Algorithm:

1. Initialize $f_0(\boldsymbol{x}) = \arg\min_c \sum_{t=1}^{T} L(y_t, c)$.
2. For $m = 1, \ldots, M$:
   - 2.1 For $t = 1, \ldots, T$, compute:

$$r_{tm} = -\left[\frac{\partial L(y_t, f(\boldsymbol{x}_t))}{\partial f(\boldsymbol{x}_t)}\right]_{f=f_{m-1}}$$

   - 2.2 Fit a regression tree for r_{tm} giving terminal regions (leaves) R_{jm}, $j = 1, \ldots, K_m$.
   - 2.3 For $j = 1, 2, \ldots, K_m$, compute

$$c_{jm} = \arg\min_c \sum_{x_t \in R_{jm}} L(y_t, f_{m-1}(x_t) + c)$$

   - 2.4 Update $f_m(\boldsymbol{x}) = f_{m-1}(\boldsymbol{x}) + \nu \sum_{j=1}^{K_m} c_{jm} I(\boldsymbol{x} \in R_{jm})$
3. Output: $\widehat{f}(\boldsymbol{x}) = f_M(\boldsymbol{x})$

Boosting Regression Trees

- Note that for square loss:

$$\frac{\partial L(y_t, f(\boldsymbol{x}_t))}{\partial f(\boldsymbol{x}_t)} = y_t - f(\boldsymbol{x}_t)$$

- What is the right size of the trees (K_m)?
  - Simple strategy: $K_m = K, \forall\, m$, K small.
  - The larger is K, the larger is the interaction among different variables. For example, if $K = 2$ there is no interaction.
  - Typically, in empirical works $4 \geq K \leq 8$.
- What are the values of M and ν?
  - ν $(0 < \nu < 1)$ is the learning rate.
  - Smaller values of ν (more shrinkage) result in larger training risk for the same number of iterations M.
  - Both ν and M control the prediction risk in the training data and do not operate independently.
  - Typically, ν is small $(\nu < 0.1)$ and M is chosen by *early stopping*.

Regression Trees

Boosting and Relative Importance of Predictor Variables

- Measure of relative importance of the ith covariate for a single tree $\mathcal{T}$:

$$\mathcal{I}_i(\mathcal{T}) = \sum_{j=1}^{N} \widehat{i}^2(s_j = i),$$

  where $N = K - 1$ is the number of internal nodes and $\widehat{i}^2(\cdot)$ is the estimated improvement in squared error risk over that for a constant fit over the entire region.

- For boosted trees:

$$\mathcal{I}_i = \sum_{m=1}^{M} \mathcal{I}_i(\mathcal{T}_m).$$

Boosting Regression Trees

Example: California Housing

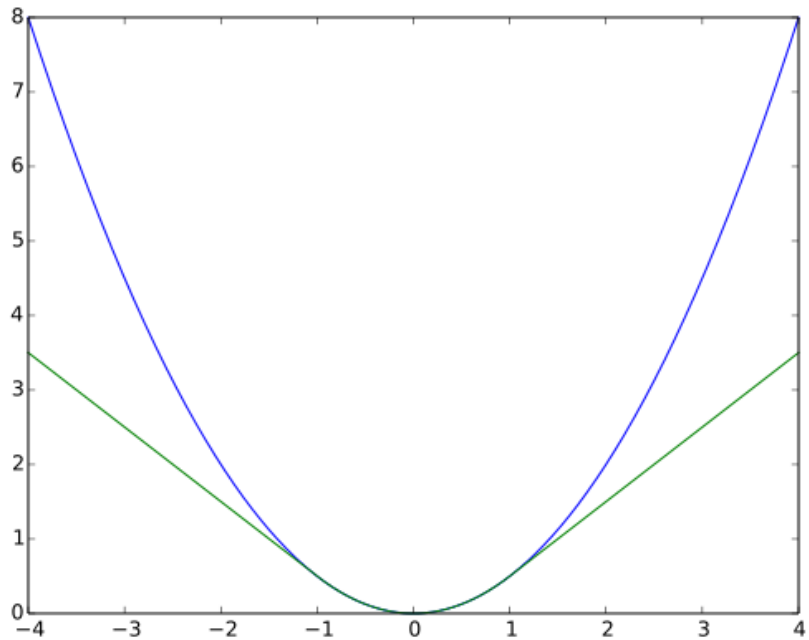
- Reference: Hastie, Tibshirani and Friedman (2009), Section 10.14.1 and Pace and Barry (1997,Stat&Prob Letters)
- Data is available form Carnegie-Mellon *Statlib* repository.
- The dataset consists of aggregated data from $20,460$ neighborhood (1990 census block groups) in California.
- The dependent variable y is the median house value in each neighborhood measures in units of $\$100,000$.
- Predictors are demographic variables such as median income (MedInc), housing density as reflected by the number of houses (House), and average occupancy in each house (AveOccup); location of each neighborhood (longitude and latitute); and several quantities reflecting the properties of the houses in the neighborhood such as average number of room AveRooms and bedrooms (AvgBedrms). Total of eight predictors, all numeric.

Boosting Regression Trees

Example: California Housing

- $K = 6$ and $\nu = 0.1$.
- Huber loss to control for outliers:

$$L(y_t, f(\boldsymbol{x}_t)) = \begin{cases} \frac{1}{2}\left[y_t - f(\boldsymbol{x}_t)\right]^2 & \text{if } |y_t - f(\boldsymbol{x}_t)| \leq \delta \\ \delta|y_t - f(\boldsymbol{x}_t)| - \frac{1}{2}\delta^2 & \text{otherwise.} \end{cases}$$



Boosting Regression Trees

Example: California Housing

- For the Huber loss:

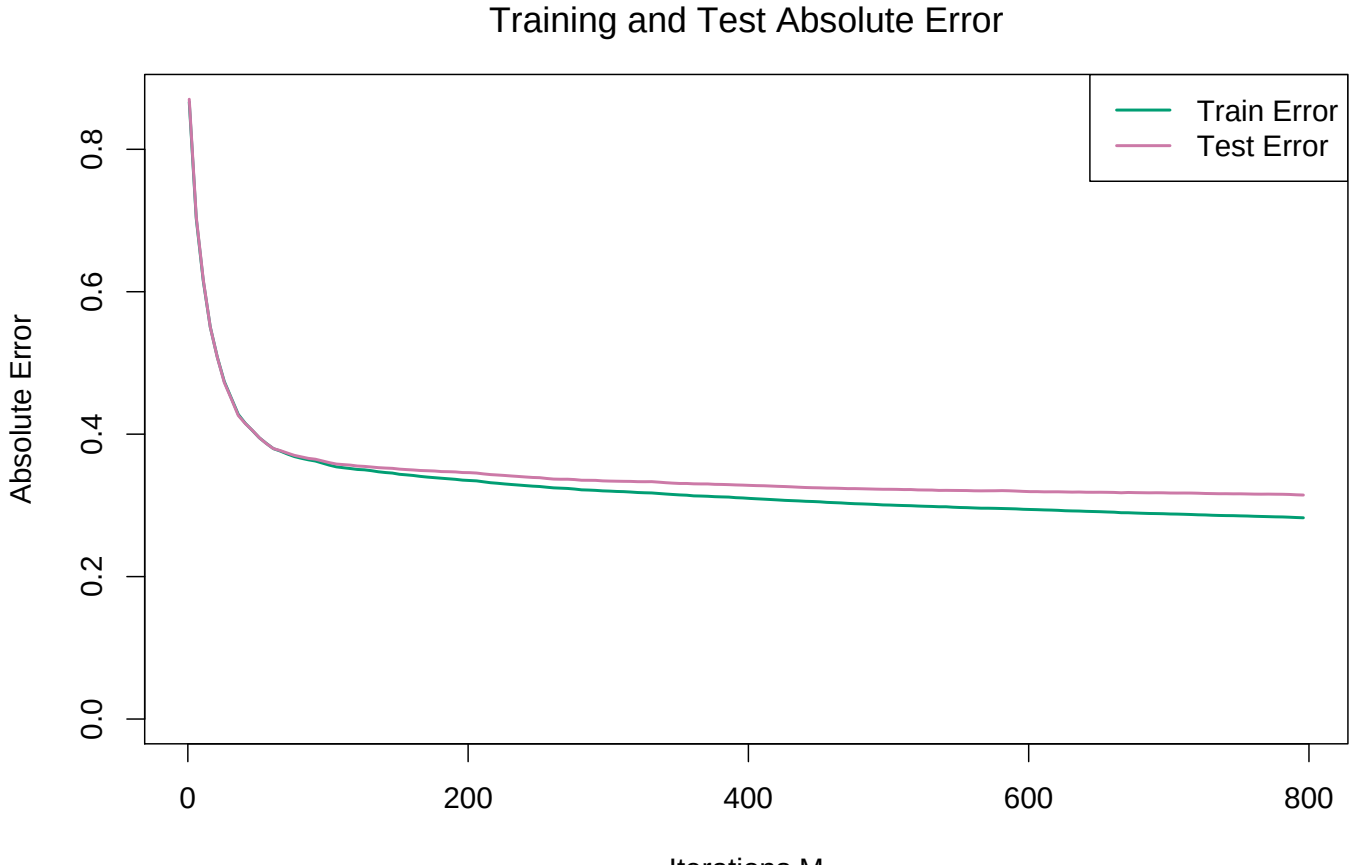
$$\frac{\partial L(y_t, f(\boldsymbol{x}_t))}{\partial f(\boldsymbol{x}_t)} = \begin{cases} y_t - f(\boldsymbol{x}_t) & \text{for } |y_i - f(\boldsymbol{x}_t)| \leq \delta \\ \delta \mathsf{sign}[y_t - f(\boldsymbol{x}_t)] & \text{otherwise,} \end{cases}$$

  where $\delta = \alpha - \text{quantile}[|y_i - f(\boldsymbol{x}_t)|]$.

Boosting Regression Trees

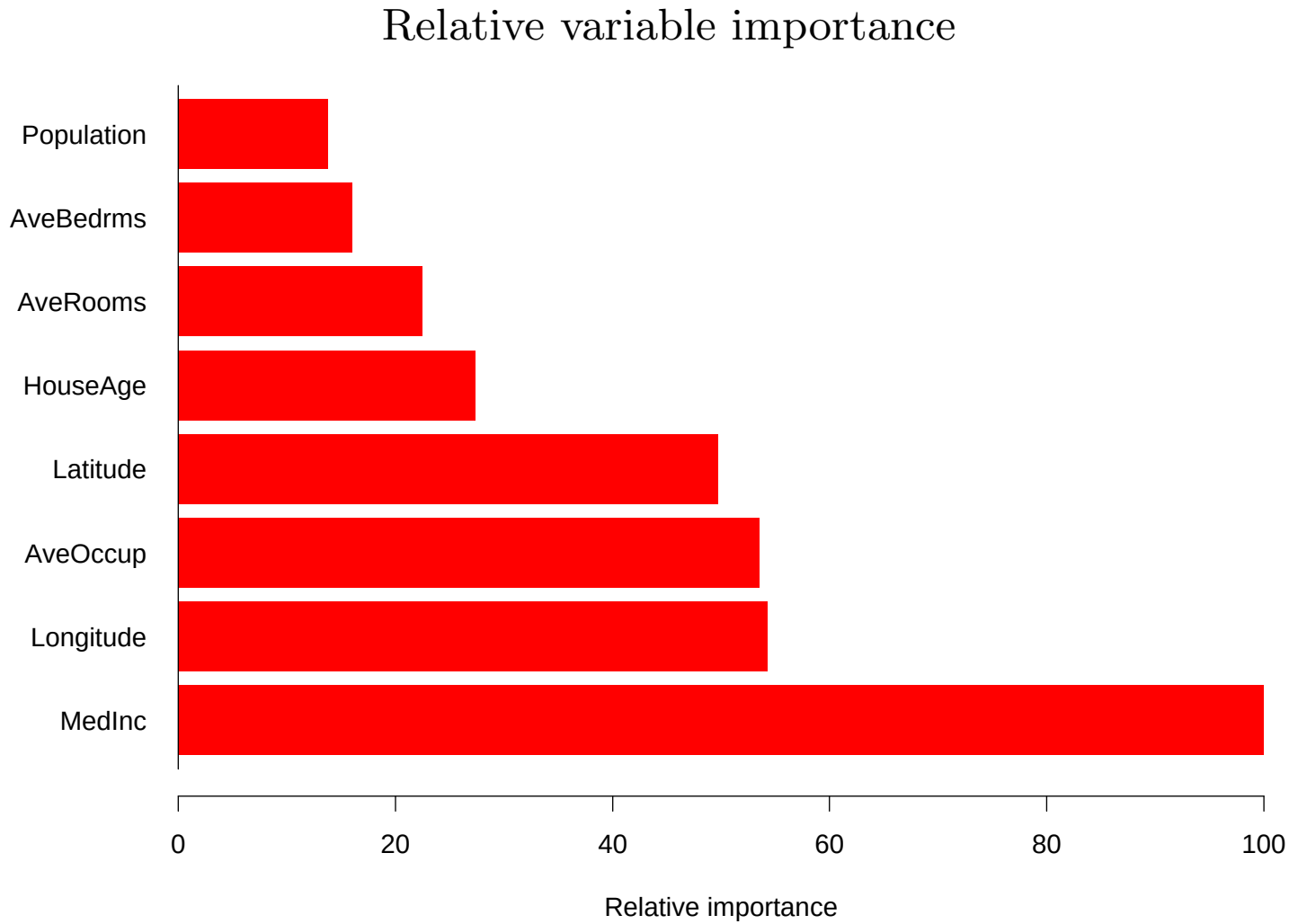
Example: California Housing

Average-absolute error as a function of iterations



Boosting Regression Trees

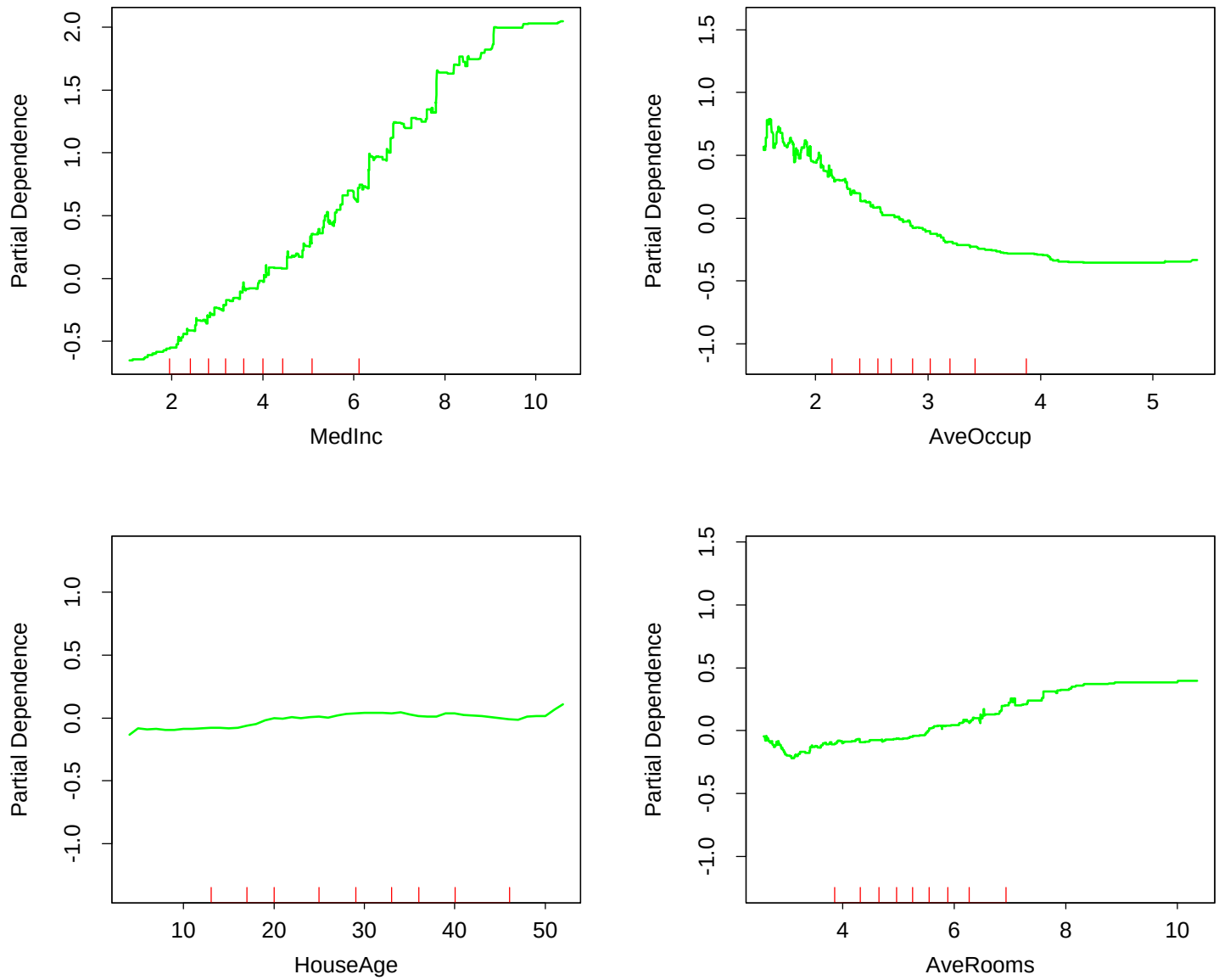
Example: California Housing



Boosting Regression Trees

Example: California Housing

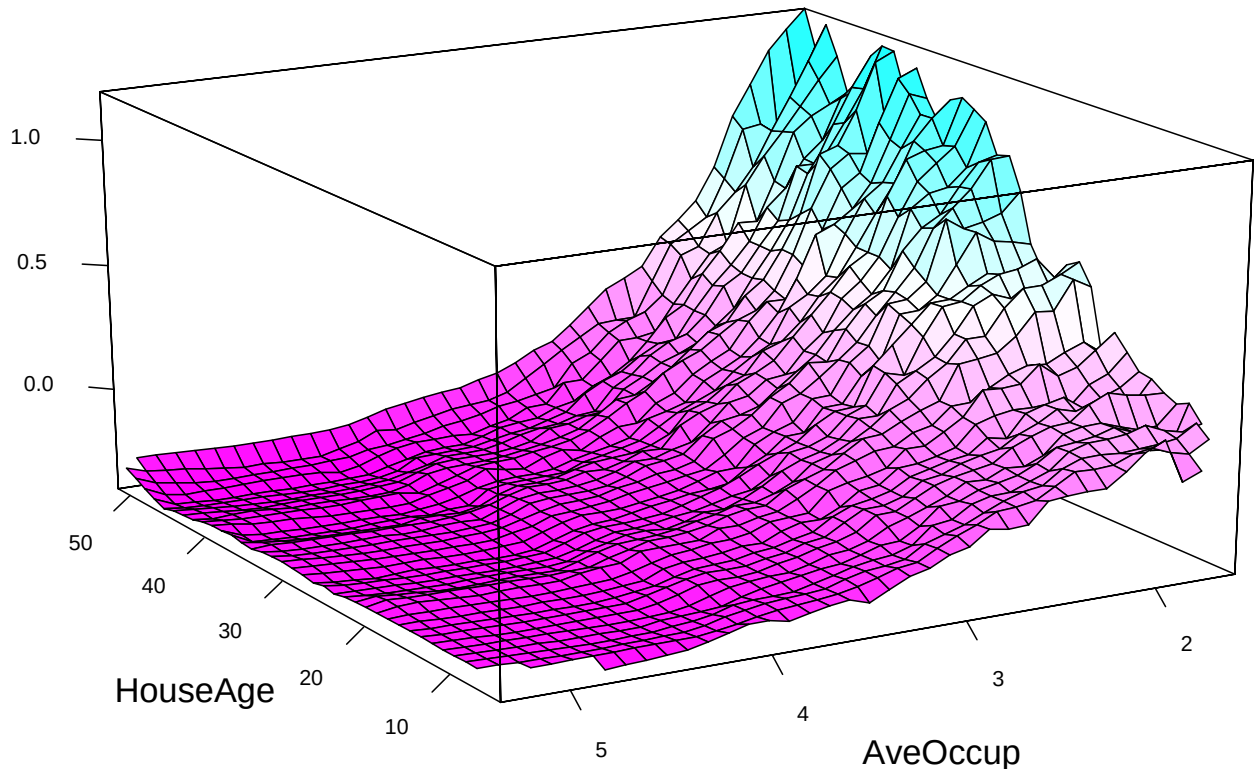
Partial dependence of housing value on nonlocation variables



Boosting Regression Trees

Example: California Housing

Partial dependence of house value on median age and average occupancy



Boosting Regression Trees

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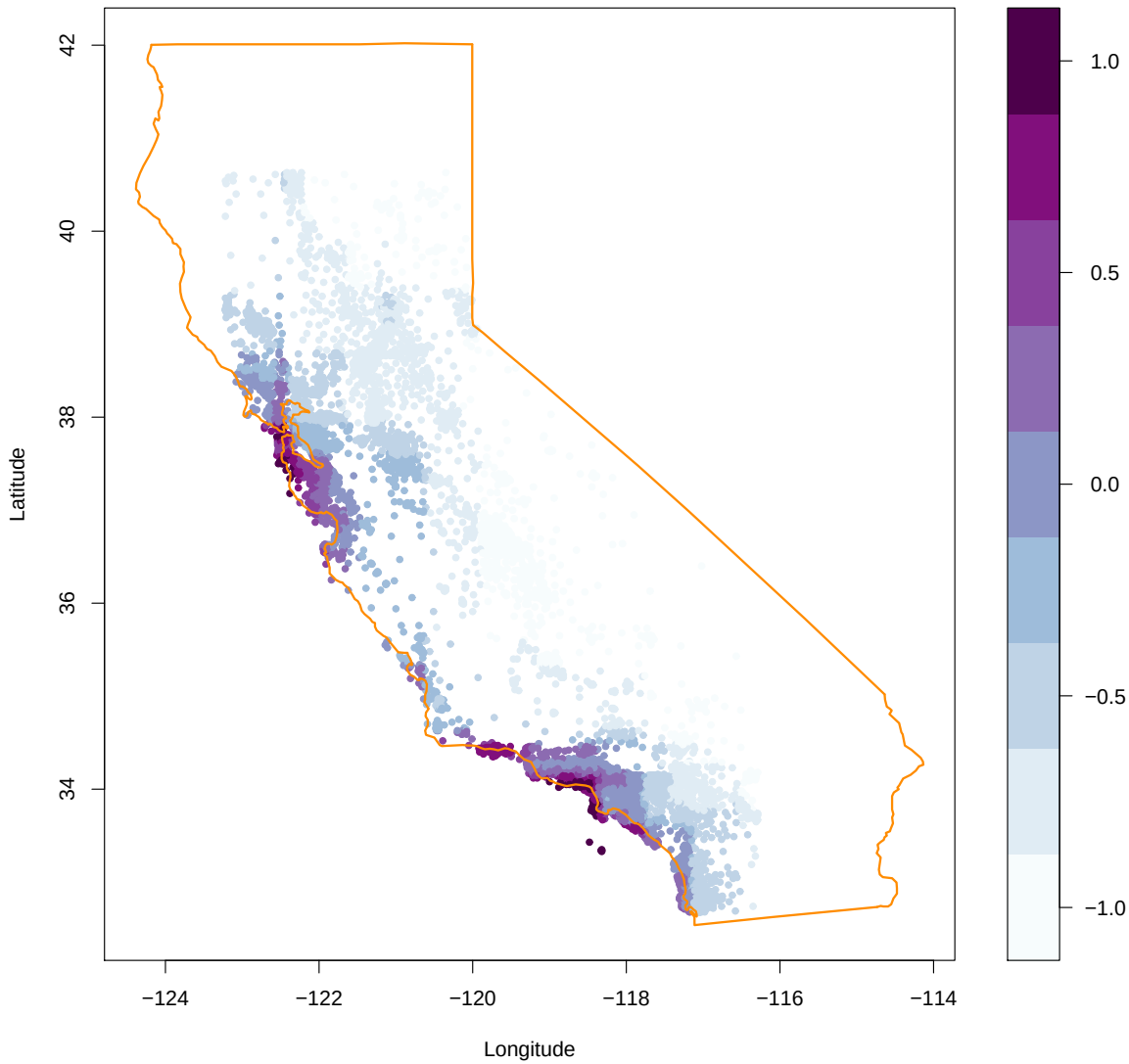

Bagging Regression Trees: Random Forests

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$$\boldsymbol{Z}_t = (y_t, \boldsymbol{x}_t'), \quad t = 1, \ldots, T.$$

- Let

$$\widehat{\theta}_T(\boldsymbol{x}) = h_T(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_T)(\boldsymbol{x})$$

be an estimator of $\mathbb{E}(y|\boldsymbol{x})$.

What is Bagging?

The Bagging Algorithm

1. Construct a bootstrap sample

$$\boldsymbol{Z}_t^* = (y_t^*, \boldsymbol{x}_t^{*'}), \quad t = 1, \ldots, T.$$

according to the empirical distribution of the pairs $(y_t, \boldsymbol{x}_t)$, $t = 1, \ldots, T$.

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2. Compute the bootstrapped predictor $\widehat{\theta}_T^*(\boldsymbol{x})$ by the plug-in principle; that is,

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3. The bagged predictor is

$$\widehat{\theta}_T^B(\boldsymbol{x}) = \mathbb{E}^* \left[\widehat{\theta}_T^*(\boldsymbol{x}) \right]$$

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- There can be a drastic variance reduction if the predictor is unstable.

What is Bagging?

Stability of a predictor

A statistic

$$\widehat{\theta}_T(\boldsymbol{x}) = h_T(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_T)(\boldsymbol{x})$$

is called stable at $\boldsymbol{x}$ if

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as $n \longrightarrow \infty$ for some fixed value $\theta(\boldsymbol{x})$.

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- Subset model selection via testing creates unstable predictors.
- **Bagging reduces the variance of unstable predictors**.

Random Forests

Bagging and Random Forests

Algorithm:

1. For $b = 1, \ldots, B$:
   - 1.1 Draw a bootstrap sample of size T from the original data (sampling with replacement).
   - 1.2 For each sample, estimate a tree $\mathcal{T}_b$, using as potential split variables a subset of q out of the p original variables randomly chosen. The tree should grow until the minimum number of observations in each leaf is reached. **No pruning**.
2. The final prediction is given as:

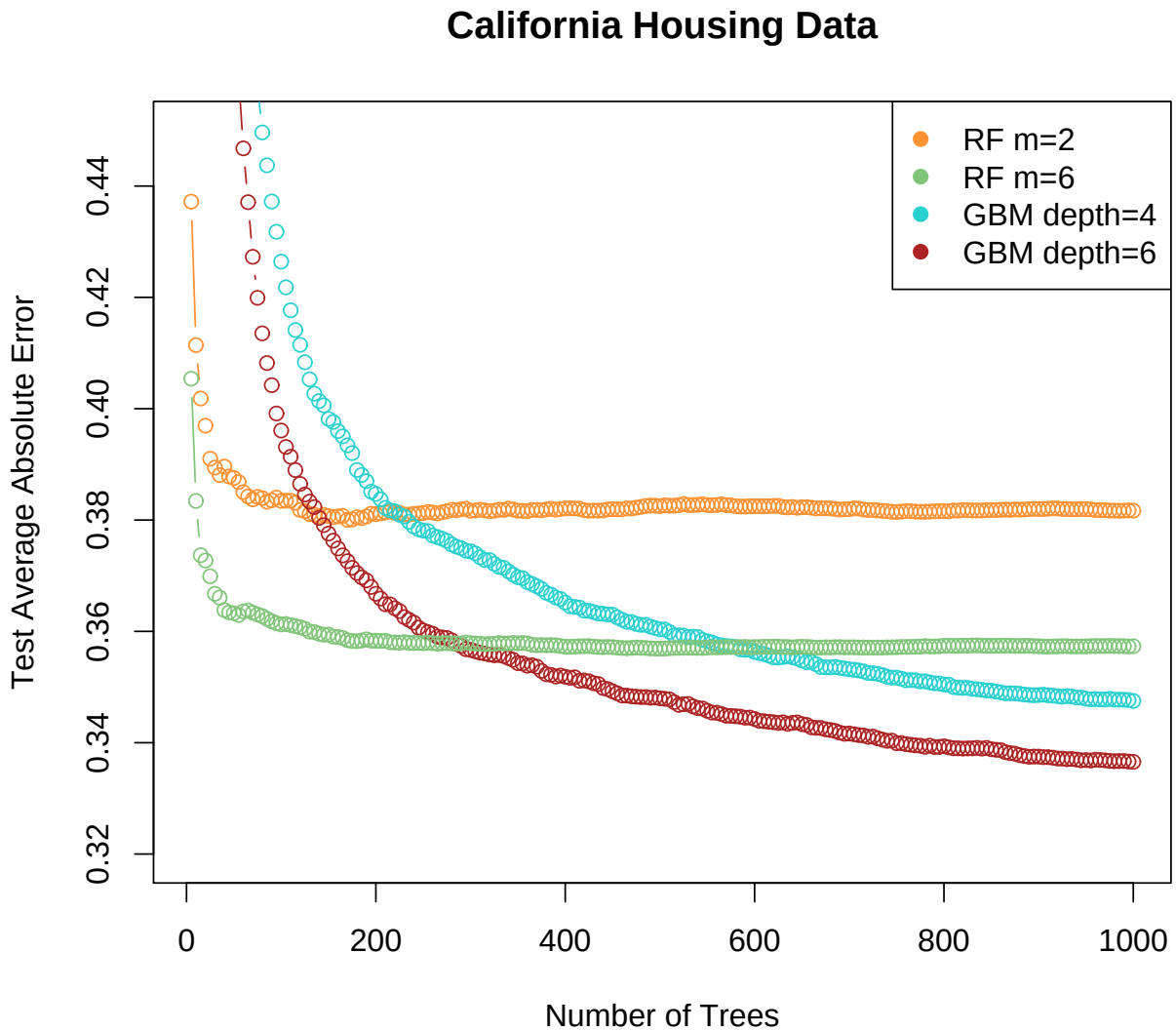
$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{T}_b(\boldsymbol{x}),$$

3. B can be monitored by the *Out-of-the-Bag* error: for each observation $\boldsymbol{z}_t = (y_t, \boldsymbol{x}_t')'$, construct its random forest predictor by averaging only those trees corresponding to bootstrap samples in which $\boldsymbol{z}_t$ did not appear.

Random Forests

Example: California Housing

Test average absolute error as a function of the number of trees



Sieves and Neural Networks

Sieve Spaces

Motivation

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- Sieve estimators have several advantages:
  1. Computational easiness
  2. Easy to impose restrictions
  3. Much easier solution under endogeneity

Sieve Spaces

Definition

- Consider the general nonlinear, semi-parametric model:

$$\begin{aligned} y_i &= m_0(\boldsymbol{x}_i) + u_i \\ &= \underbrace{\boldsymbol{\beta}_0' \boldsymbol{x}_i}_{\text{parametric component}} + \underbrace{h_0(\boldsymbol{x}_i)}_{\text{nonparametric component}} + u_i, \end{aligned}$$

where $\{u_i\}$, $i = 1, \ldots, n$ is a sequence of random disturbances and $\boldsymbol{x}_i \in \mathbb{R}^p$ is a vector of covariates.

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- Let's assume for now that $\mathbb{E}(u_i|\boldsymbol{x}_i) = 0$ and $\mathbb{E}(u_i^2|\boldsymbol{x_i}) = \sigma^2(\boldsymbol{x}_i) < \infty$, for all $i = 1, \ldots n$.

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- The goal is to estimate the vector of parameters $\boldsymbol{\theta}_0 = (\boldsymbol{\beta}_0', h_0)' \in \mathcal{B} \times \mathcal{H} \equiv \Theta$, where $\mathcal{B}$ denotes a finite-dimensional compact parameter space and $\mathcal{H}$ is an infinite-dimensional parameter space.

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- A typical semi-nonparametric econometric model specifies that there is a population criterion function $Q : \Theta \to \mathbb{R}$ which is uniquely maximized (or minimized) at a (pseudo-)true parameter $\boldsymbol{\theta}_0$.
- When Θ is infinite-dimensional and possibly not compact with respect to the (pseudo-)metric d, maximizing and empirical criterion function $\widehat{Q}_n$ over Θ may not be well-defined.

Sieve Spaces

Definition

- We say that the optimization problem is well-posed if for sequences $\{\boldsymbol{\theta}_k\} \in \Theta$ such that $Q(\boldsymbol{\theta}_0) - Q(\theta_k) \to 0$, then $d(\boldsymbol{\theta}_0, \boldsymbol{\theta}_k) \to 0$.

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Sieve Spaces

Definition

Key Idea

- The method of sieves provides a general approach to resolve difficulties associated with maximizing $\widehat{Q}_n$ over Θ by maximizing $\widehat{Q}_n$ over a sequence of approximating spaces Θ_n, called *Sieves*.

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- These spaces are less complex than Θ but are **dense** in Θ.
- Popular sieves are typically compact, nondecreasing ($\Theta_n \subseteq \Theta_{n+1} \subseteq \cdots \subseteq \Theta$) and are such that for any element $\boldsymbol{\theta} \in \Theta$ there exists as element $\pi_n \boldsymbol{\theta} \in \Theta_n$ satisfying $d(\boldsymbol{\theta}, \pi_n \boldsymbol{\theta}) \to 0$ as $n \to \infty$, where the notation π_n can be regarded as a projection mapping from Θ to Θ_n.

Sieve Spaces

Definition

- The *approximate sieve extremum estimate*, denoted by $\widehat{\boldsymbol{\theta}}_n$, is defined as an approximate maximizer of $\widehat{Q}_n(\boldsymbol{\theta})$ over the sieve space Θ_n:

$$\widehat{Q}_n(\widehat{\boldsymbol{\theta}}_n) \geq \sup_{\boldsymbol{\theta} \in \Theta_n} \widehat{Q}_n(\boldsymbol{\theta}) - O_p(\eta_n),$$

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  with $\eta_n \to 0$ as $n \to \infty$.
- When $\eta_n = 0$ we call the estimator the *exact sieve extremum estimate*.

Sigmoid Artificial Neural Network

The single hidden layer Sigmoid Artificial Neural Network (sANN) sieve is defined as

$$\mathsf{sANN}(J_n) = \left\{ \gamma_0 + \sum_{j=1}^{J_n} \alpha_j S(\boldsymbol{\gamma}_j' \boldsymbol{x} + \gamma_{0,j}) : \boldsymbol{\gamma}_j \in \mathbb{R}^p, \alpha_j, \gamma_0, \gamma_{0,j} \in \mathbb{R} \right\}.$$

- $S : \mathbb{R} \to R$ is a sigmoid "activation" function, i.e., a bounded nondecreasing function such that $\lim_{u \to -\infty} S(u) = 0$ and $\lim_{u \to \infty} S(u) = 1$.

Nonlinear Sieve Spaces

Single Hidden Layer Neural Networks

- Some popular sigmoid functions:
  - Heaviside: $S(u) = 1(u \geq 0)$;
  - Logistic: $S(u) = 1/[1 + \exp(-u)]$;
  - Hyperbolic tangent:
    $S(u) = [\exp(u) - \exp(-u)]/[\exp(u) + \exp(-u)]$;
  - Gaussian sigmoid: $S(u) = (2\pi)^{-1/2} \int_{-\infty}^{u} \exp(-y^2/2)dy$;
  - Cosine squasher:
    $S(u) = [1 + \cos(u + 3\pi/2)]/21(|u| \leq \pi/2) + 1(u > \pi/2)$.
  - ReLU: $S(u) = u\mathsf{I}(u > 0)$

Empirical Example: Equity Premium Forecasting with ML Methods

Gu, Shihao, Bryan Kelly and Dacheng Xiu (2018). *Empirical Asset Pricing via Machine Learning*. Working paper available at SSRN id 3159577.

Main Idea

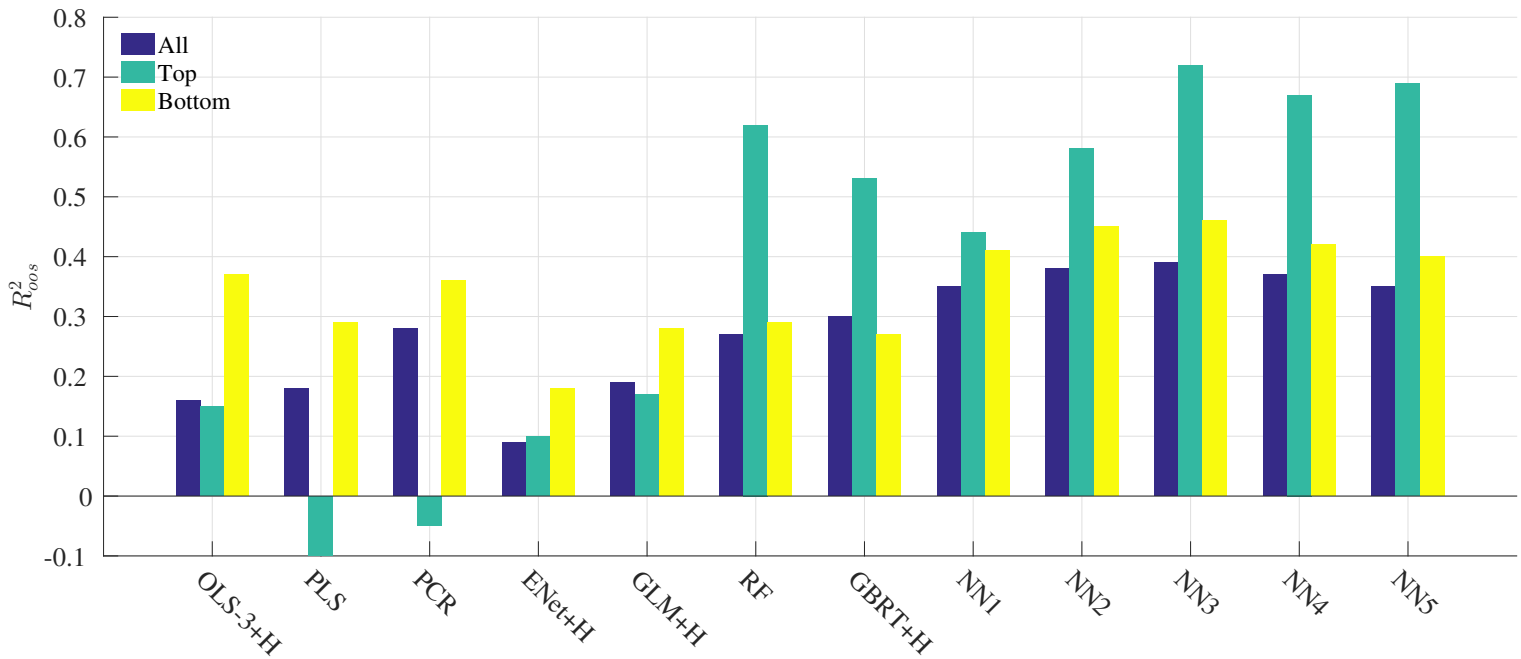
- Model:

$$\boxed{\begin{aligned} r_{t,t+1} &= \mathbb{E}_t(r_{i,t+1}) + \epsilon_{i,t+1} \\ \mathbb{E}_t(r_{i,t+1}) &= g(\boldsymbol{z}_{i,t}) \end{aligned}}$$

  where $\boldsymbol{z}_{it}$ is a large vector of predictors:
  - 91 firm characteristics (61 of which are updated annually, 13 updated quarterly and 20 updated monthly);
  - 74 industry dummies
  - 8 macroeconomic predictors
  - Interactions between macro factors and firm characteristics.
- 30,000 stocks over a sample starting in March 1957 and ending in December 2016 (60 years).
- ML methods: linear regression, restricted linear regression (FF factors), partial least squares (PLS), principal component regression (PCR), generalized linear model (GLM), random forest (RF), Boosted Trees (GBRT), neural networks with 1 to 5 layers (NN1–NN5).

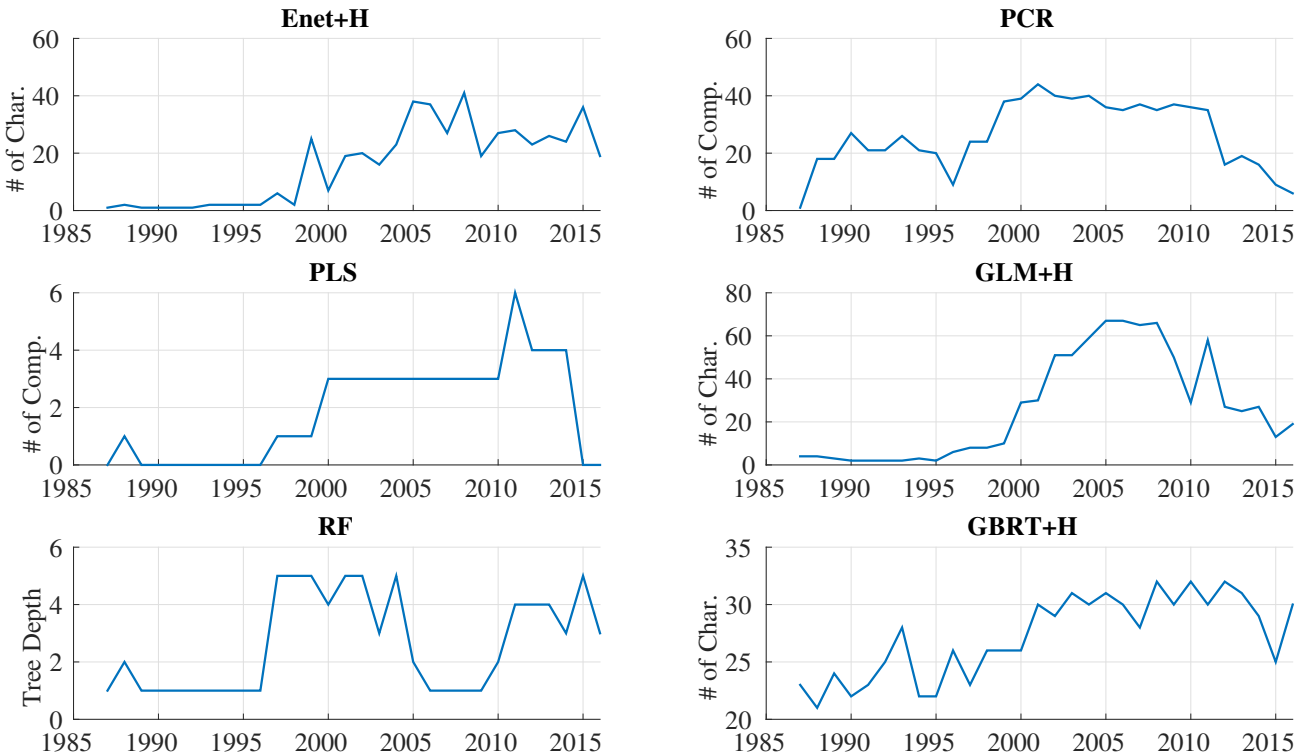
Results

	OLS +H	OLS-3 +H	PLS	PCR	ENet +H	GLM +H	RF	GBRT +H	NN1	NN2	NN3	NN4	NN5
All	-4.60	0.16	0.18	0.28	0.09	0.19	0.27	0.30	0.35	0.38	0.39	0.37	0.35
Top 1000	-14.21	0.15	-0.10	-0.05	0.10	0.17	0.62	0.53	0.44	0.58	0.72	0.67	0.69
Bottom 1000	-2.13	0.37	0.29	0.36	0.18	0.28	0.29	0.27	0.41	0.45	0.46	0.42	0.40



Note: In this table, we report monthly R^2_{oos} for the entire panel of stocks using OLS with all variables (OLS), OLS using only size, book-to-market, and momentum (OLS-3), PLS, PCR, elastic net (ENet), generalize linear model (GLM), random forest (RF), gradient boosted regression trees (GBRT), and neural networks with one to five layers (NN1–NN5). "+H" indicates the use of Huber loss instead of the l_2 loss. We also report these R^2_{oos} within subsamples that include only the top 1,000 stocks or bottom 1,000 stocks by market value. The lower panel provides a visual comparison of the R^2_{oos} statistics in the table (omitting OLS due to its large negative values).

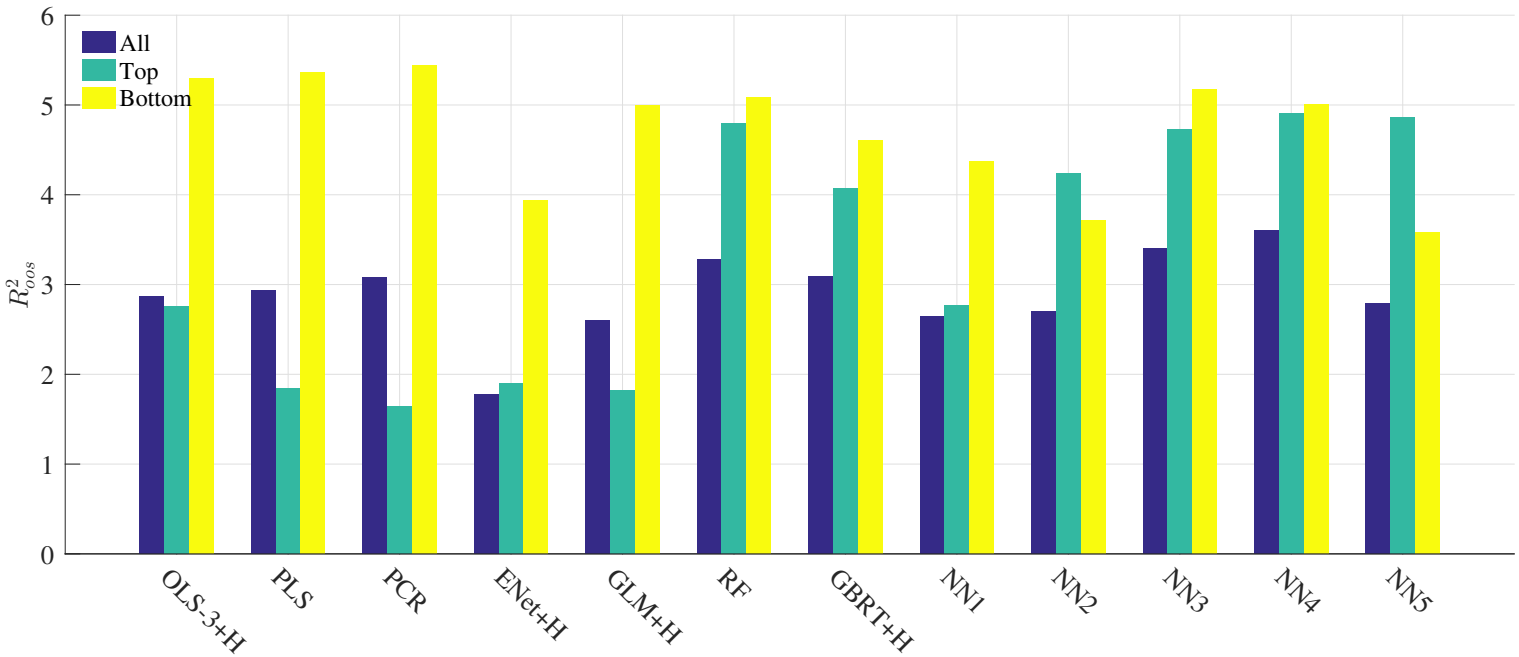
Results



Note: This figure demonstrates the model complexity for elastic net (ENet), PCR, PLS, generalized linear model with group lasso (GLM), random forest (RF) and gradient boosted regression trees (GBRT) in each training sample of our 30-year recursive out-of-sample analysis. For ENet and GLM we report the number of features selected to have non-zero coefficients; for PCR and PLS we report the number of selected components; for RF we report the average tree depth; and for GBRT we report the number of distinct characteristics entering into the trees.

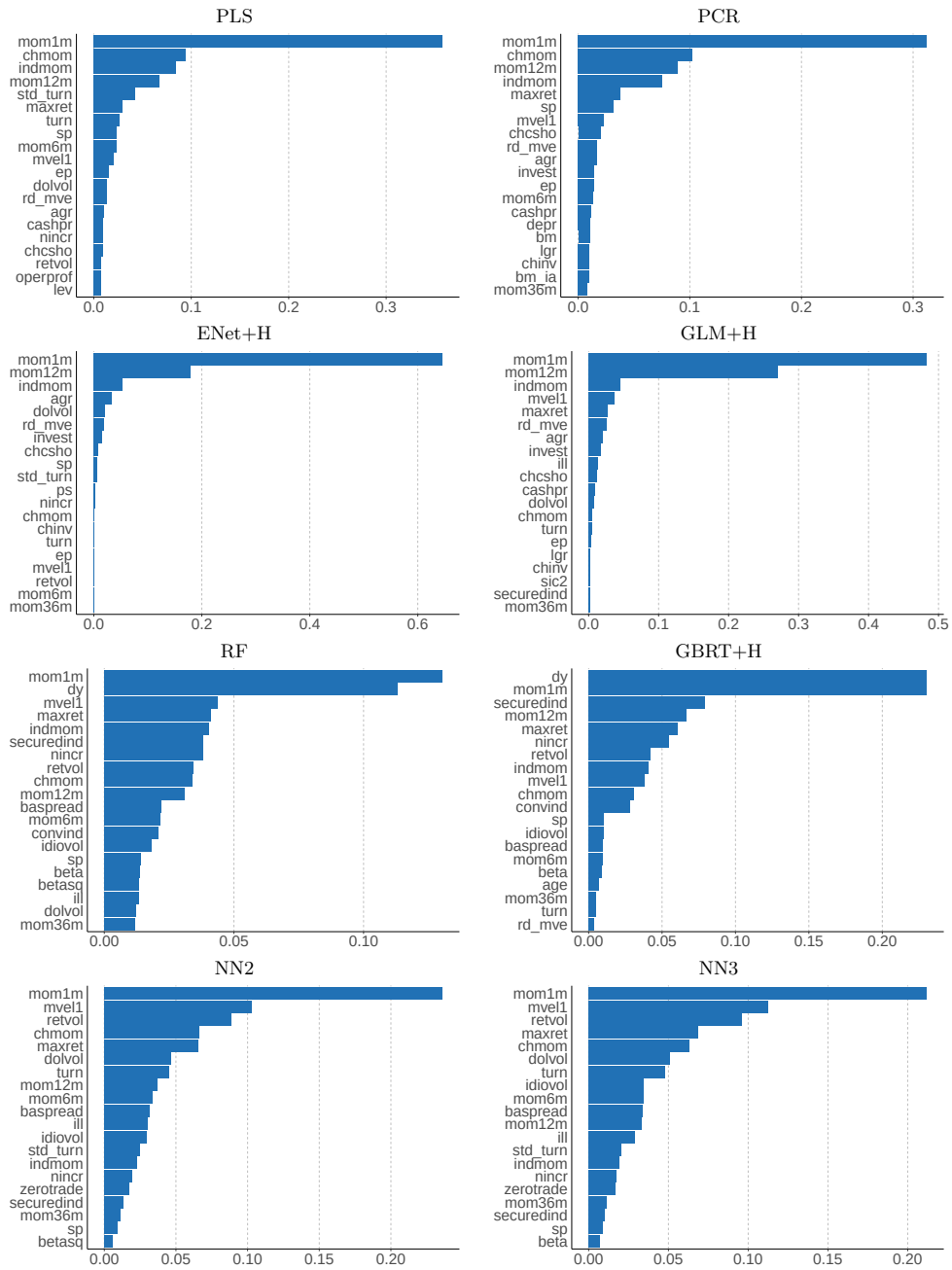
Results

	OLS +H	OLS-3 +H	PLS	PCR	ENet +H	GLM +H	RF	GBRT +H	NN1	NN2	NN3	NN4	NN5
All	-34.86	2.87	2.93	3.08	1.78	2.60	3.28	3.09	2.64	2.70	3.40	3.60	2.79
Top	-54.86	2.77	1.84	1.64	1.90	1.82	4.80	4.07	2.77	4.24	4.73	4.91	4.86
Bottom	-19.22	5.30	5.36	5.44	3.94	5.00	5.08	4.61	4.37	3.72	5.17	5.01	3.58



Note: Annual return forecasting R^2_{oos} (see Table 1 notes).

Results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mom12m
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