Extracting Stochastic Discount Factors from Data

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1. Information Frontiers (generalizing the minimum variance frontiers) and tests of dynamic asset pricing models
   Almeida and Garcia (2017), *Economic Implications of Nonlinear Pricing Kernels*, Management Science.
2. Measures of model misspecification
   Almeida and Garcia (2012), *Assessing misspecified asset pricing models with empirical likelihood estimators*, Journal of Econometrics
3. Evaluation of managed funds performance
   Almeida, Ardison and Garcia (2018), *Nonparametric Assessment of Hedge Fund Performance*, Working Paper.
4. Estimating a tail risk measure
   Almeida, Ardison, Garcia and Vicente (2017), *Nonparametric Tail Risk, Stock Returns, and the Macroeconomy*, Journal of Financial Econometrics

- The central idea of modern finance is that prices are generated by expected discounted payoffs

$$p_t^i = E_t(m_{t+1} x_{t+1}^i)$$

- The discount factor m_{t+1} is equal to the growth in the marginal value of wealth

$$m_{t+1} = \frac{VW(t+1)}{VW(t)}.$$

- The traditional theories of finance, CAPM, ICAPM, and APT, measure the marginal utility of wealth by the behavior of large portfolios of assets.
  - ✓ CAPM: return on the market portfolio.
  - ✓ Multifactor models: returns on multiple portfolios.
- To make the link between the real economy and financial markets, we measure the growth in marginal utility of wealth by the growth in consumption (Consumption CAPM).
  - ✓ Idea that consumption is the payoff on the market portfolio.

- In a linear factor model, asset returns are described by the following process:

$$R_{it} = a_i + b_i' \; f_t + \varepsilon_{it}$$

$$E\left[\varepsilon_{it} \mid f_t\right] = 0$$

R_{it} return at time t for asset i

a_i intercept of the factor model

b_i $(K \times 1)$ vector of factor sensitivities for asset i

f_t a $(K \times 1\ )$ vector of common factor realizations at time t

ε_{it} disturbance term.

- The SDF is of the form:

$$m_{t+1} = \alpha + \beta' f_{t+1}$$

- Asset i can be an individual security, a portfolio or a managed fund.
- Factors can be observed (returns on some portfolios or macroeconomic variables) or latent (obtained by some statistical procedure).

Consumption Capital Asset Pricing Model (CCAPM)

$$Max_{(C_t, w_{i,t+1})} E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}), i = 1, ..., n$$

under the constraint:

$$C_t + \sum_{i=1}^{n} p_{it} w_{i,t+1} \leq \sum_{i=1}^{n} (p_{it} + D_{it}) w_{i,t} + y_t$$

- One consumption good, infinite horizon, additive and time separable utility
- p_{it} = price of asset i at time t; D_{it} = dividend paid on asset i at t, beginning of period; w_{it} = units of asset i held at beginning of period t; y_t = labor income exogenous at time t.
- Fist-order condition:

$$U'(C_t) = \beta E_t \left[R_{i,t+1} U'(C_{t+1}) \right]$$

Then: $m_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}$. With isoelastic utility function: $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$

$$E_t \left[R_{i,t+1} \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] = 1 \quad (1)$$

- The asset pricing model predicts:

$$E(p_t) = E[m(data_{t+1}, parameters)x_{t+1}]$$

- A natural way to check this prediction is to examine sample averages:

$$\frac{1}{T}\sum_{t=1}^{T} p_t \text{ and } \frac{1}{T}\sum_{t=1}^{T}[m(data_{t+1}, parameters)x_{t+1}]$$

- The Generalized Method of Moments (GMM) estimates the parameters by making the sample averages as close to each other as possible.
- GMM evaluates the model by looking at how close the sample averages of price and discounted payoff are to each other (how small the pricing errors are).

- Other methods such as maximum likelihood impose stricter restrictions on distributions.
- For linear models, by far the most common in empirical asset pricing, regression-based methods (cross-sectional and time-series) are the standard.
- One important remark: all models lead to pricing errors (another way of saying all models are misspecified).

Existence of SDFs with weaker assumptions

- From $p = E(mx)$ can we always find a SDF without assuming all the structure of investors, utility functions, complete markets, linear relations, and so forth?
- **Two weaker restrictions**: the *law of one price* and the *absence of arbitrage*, help us in this direction.
- *Law of one price*: If two portfolios have the same payoffs (in every state of nature), then they must have the same price.
- **A first theorem** states that there is a discount factor that prices all the payoffs by $p = E(mx)$ if and only if the law of price holds.
- *Absence of arbitrage*: If payoff A is always at least as good as payoff B and sometimes better then the price of A must be greater than the price of B.
- A **second theorem** states that there is a *positive* discount factor that prices all the payoffs by $p = E(mx)$ if and only if there are no arbitrage opportunities and the law of one price holds.
- All that is required from investors is that they do not leave law of one price violations or arbitrage opportunities on the table.

- **These theorems** say that there exists a positive discount factor ($m > 0$) but they do not say that it is unique.
- **These theorems** say that there exists a positive discount factor ($m > 0$) but they do not say that every discount factor must be positive (in every state of nature).

- Given a K-dimensional set of benchmark factors' excess returns (equity, bonds, options, etc...) $\underline{R}^{e,f}$ and a risk-free rate R_F, any candidate SDF m must correctly price:
  - The factors' excess returns: $E[m.\underline{R}^{e,f}] = 0_K$.
  - and the risk-free asset: $E[m] = \frac{1}{R_F}$ (here we assume $R_F = 1$).

- If markets are incomplete, and LOOP holds, an infinity of possible candidates for m exists.
- If, in addition, there is no (in-sample) arbitrage, at least one strictly positive m exists.

- **Fundamental Question: How to empirically identify strictly positive SDF candidates m in a meaningful way?**

- HJ derive a SDF m that is a linear combination of a set of basis asset returns R and that obeys: $E[mR] = 1$.
- They look for a minimum-variance m since the mean-variance frontier of all discount factors that price a given set of assets is related to the mean-variance frontier of asset excess returns by:

$$\frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)}$$

- which implies a nice duality:

$$\underbrace{\min}_{\text{all } m \text{ that price the basis assets}} \frac{\sigma(m)}{E(m)} = \underbrace{\max}_{\text{all excess returns on basis assets}} \frac{|E(R^e)|}{\sigma(R^e)}$$

- They derive SDFs without positivity constraint (the SDF may be negative in some states of nature) or with positivity constraint (the SDF is set to zero in states of nature where it is negative).

A More General Class of Admissible SDFs

- Why not consider all strictly positive admissible SDFs?
  - ✓ Set is too large. There is no empirical guideline on how to identify all strictly positive SDFs.

However, a possible way to select specific strictly positive SDFs is provided by Almeida and Garcia (Manag. Sci., 2017):

Given a convex discrepancy (penalty) function $\phi(m)$, we define the minimum discrepancy problem as:

$$
\begin{aligned}
m_{MD} = \arg\min_{m>0} \quad & E[\phi(m)] \\
\text{subject to} \quad & E[m.\underline{R}^{e,f}] = 0_K \\
& E[m] = 1
\end{aligned}
\tag{1}
$$

where m is the "best" available candidate in the discrepancy sense.

Estimating SDFs: Sample Version

- Assume that factors' returns have realizations $\underline{R}_t^{e,f}$.
- Now, any SDF must correctly price factors' excess returns, in sample:

$$\frac{1}{T}\sum_{t=1}^{T} m_t \underline{R}_t^{e,f} = 0_K \tag{2}$$

- The sample version of the Minimum Discrepancy implied SDF is obtained by:

$$\hat{m}_{MD} = \underset{\{m_1,\ldots m_T\}}{arg\ \ min}\ \tfrac{1}{T}\textstyle\sum_{t=1}^{T}\phi(m_t),$$

$$\text{s.t. } \tfrac{1}{T}\textstyle\sum_{t=1}^{T} m_t \underline{R}_t^{e,f} = 0_K,\ \tfrac{1}{T}\textstyle\sum_{t=1}^{T} m_t = 1, m_t > 0 \forall t.$$

Solving a MD Problem in the Dual Space

It is simpler to solve the original primal discrepancy problem in the dual space (Borwein and Lewis, SICON, 1991):

$$\hat{\lambda} = \underset{\alpha \in \Re, \lambda \in \Lambda}{arg\ sup} \frac{\alpha}{R_f} - \sum_{t=1}^{T} \frac{1}{T} \phi^{*,+} \left(\alpha + \lambda' \underline{R}_t^{e,f} \right), \tag{3}$$

where $\Lambda \subseteq R^K$ and $\phi^{*,+}$ denotes the convex conjugate of ϕ restricted to the positive real line:

$$\phi^{*+}(z) = \sup_{w>0} zw - \phi(w) \tag{4}$$

Specializing the MD problem to the Cressie Read Family

- The methodology allows for any convex function ϕ.
- Almeida & Garcia (JoE, 2012, Manag. Sci. 2017) note:
- The Cressie Read family, $\phi(m) = \frac{(m)^{\gamma+1}-1}{\gamma(\gamma+1)}$, contains several special cases of interest:

1. Hansen and Jagannathan (JPE, 1991, $\gamma = 1$): “linear” discounting.
2. Empirical Likelihood ($\gamma = -1$): Bansal & Lehmann (MaDyn, 1997).
3. Exponential Tilting ($\gamma = 0$): Stutzer (JoE, 1995).
4. Hellinger ($\gamma = -0.5$): Kitamura et al. (ECTA, 2013).

The Implied SDFs under Cressie Read

- We can interpret CR dual problems as HARA utility maximizing problems:

$$\hat{\lambda}_{CR} = \underset{\lambda \in \Lambda_{CR}}{arg\ sup} \frac{1}{T} \sum_{t=1}^{T} -\frac{1}{\gamma+1} \left(1 + \gamma \lambda' \underline{R}_t^{e,f}\right)^{\left(\frac{\gamma+1}{\gamma}\right)} \qquad (5)$$

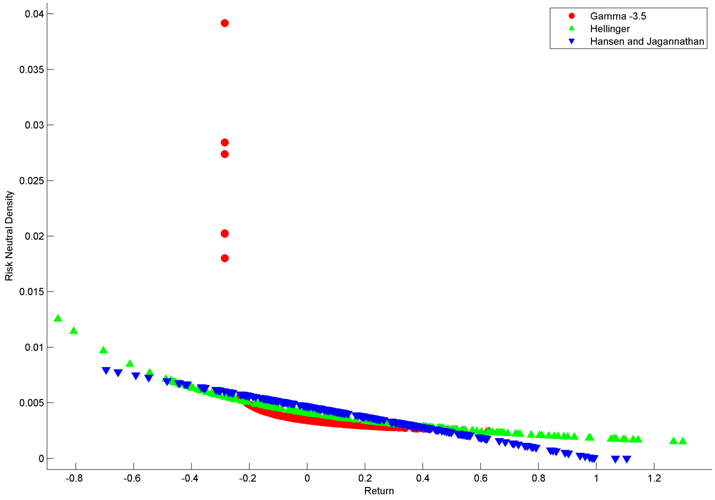
$$\Lambda_{CR} = \{\lambda \in \mathbb{R}^K | \forall t = 1, \ldots, T \ (1 + \gamma \lambda' \underline{R}_t^{e,f}) > 0\}$$

- The implied SDF is recovered via the f.o.c of problem 5:

$$\hat{m}_{MD}^t = T * \frac{(1 + \gamma \hat{\lambda}'_{CR} \underline{R}_t^{e,f})^{\frac{1}{\gamma}}}{\sum_{j=1}^{T} (1 + \gamma \hat{\lambda}'_{CR} \underline{R}_j^{e,f})^{\frac{1}{\gamma}}} \qquad (6)$$

- Important Question: How does the choice of gamma affect risk neutralization?

Direct Effect of Risk Neutralization

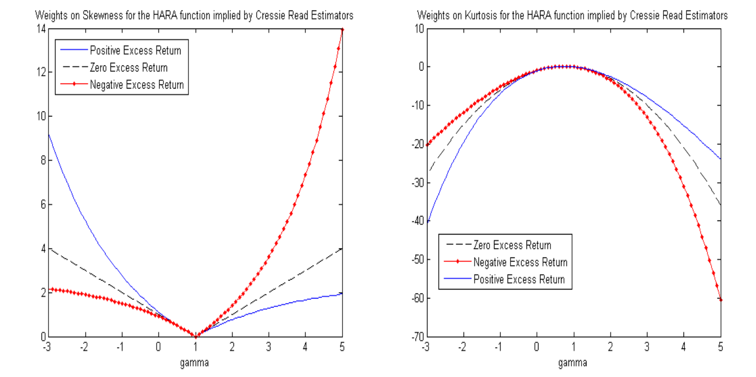


- Taylor expanding the expected value of $\phi(m) = \frac{m^{\gamma+1} - a^{\gamma+1}}{(\gamma(\gamma+1))}$ around the SDF mean a.

$$E(\phi(m)) = \frac{a^{\gamma-1}}{2} E(m-a)^2 + \frac{(\gamma-1)a^{\gamma-2}}{3!} E(m-a)^3 + \frac{(\gamma-1)(\gamma-2)a^{\gamma-3}}{4!} E(m-a)^4 + ...$$

- The weights given to skewness and kurtosis are respectively $\frac{(\gamma-1)a^{\gamma-2}}{3!}$ and $\frac{(\gamma-1)(\gamma-2)a^{\gamma-3}}{4!}$
- Plotting the weights as a function of γ.

Skewness and Kurtosis Weights in CR Discrepancies



Applications

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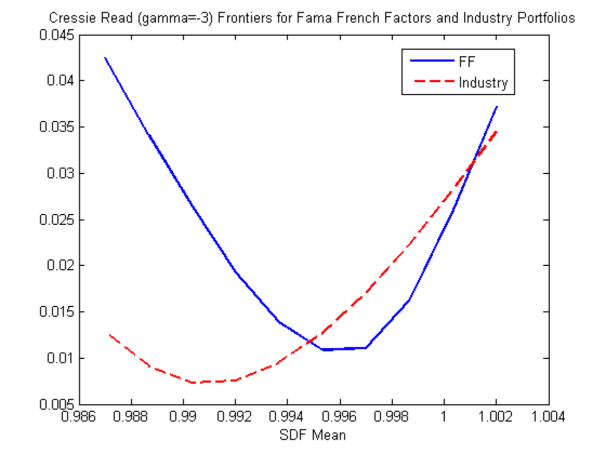
- We need to vary the mean a of the SDF to generate the frontier.
- The dual problem is written as:

$$\hat{\lambda}_{CR} = \underset{\lambda \in \Lambda_{CR}}{arg\ sup} \frac{1}{T} \sum_{i=1}^{T} -\frac{1}{\gamma+1}(1+\gamma\lambda'\left(R_i - \frac{1}{a}\right))^{(\frac{\gamma+1}{\gamma})} \quad (7)$$

- The implied SDF is recovered via the first order conditions of the problem: $\hat{m}^i_{MD} = T * a * \frac{(1+\gamma\hat{\lambda}'_{CR}\left(R_i - \frac{1}{a}\right))^{\frac{1}{\gamma}}}{\sum_{j=1}^{T}(1+\gamma\hat{\lambda}'_{CR}\left(R_j - \frac{1}{a}\right))^{\frac{1}{\gamma}}}$
- The SDF-related frontier is found by solving (7) for a grid of values for the SDF mean $A = \{a_1, a_2, ..., a_J\}$.
- The SDF-related frontier is given by:

$$I_{CR}(a_l, \gamma) = \frac{1}{T} \sum_{i=1}^{T} \frac{\hat{m}^i_{MD}(a_l)^{\gamma+1} - 1}{\gamma(\gamma+1)}, l = 1, 2, ..., J \quad (8)$$

Frontiers implied by Different Data Sets



- Barro model (2006): a disaster-like drop in aggregate consumption growth produces a large equity premium and captures other non-normal features of asset returns.

$$g_{t+1} = \eta_{t+1} + J_{t+1}$$

  η_{t+1} is the normal component $\aleph(\mu, \sigma^2)$; J_{t+1} is a Poisson mixture of normals.

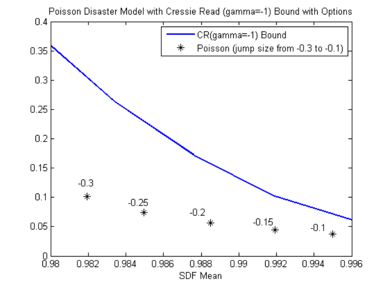
- The number-of-jumps variable j takes integer values with probabilities $e^{-\tau} \frac{\tau^j}{j!}$, where τ is the jump intensity. Conditionally on the number of jumps, J_t is normal:

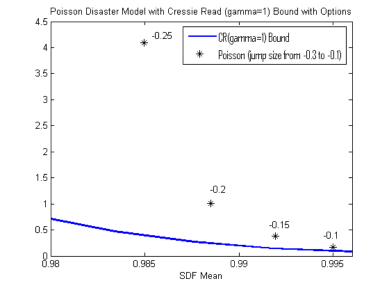
$$J_t | j \sim \aleph(j\alpha, j\lambda^2).$$

- The logarithm of the stochastic discount factor with power utility is:

$$\log m_{t+1} = \log \beta - \zeta g_{t+1}$$

  where ζ is the coefficient of relative risk aversion.


Poisson Disaster Model with Cressie Read (gamma=-1) Bound with Options
CR(gamma=-1) Bound
Poisson (jump size from -0.3 to -0.1)
-0.3
-0.25
-0.2
-0.15
-0.1
SDF Mean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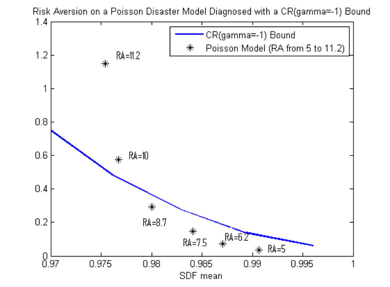
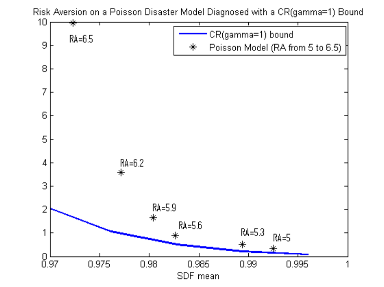

Risk Aversion on a Poisson Disaster Model Diagnosed with a CR(gamma=-1) Bound
CR(gamma=-1) Bound
Poisson Model (RA from 5 to 11.2)
RA=11.2
RA=10
RA=8.7
RA=7.5
RA=6.2
RA=5
SDF mean


Risk Aversion on a Poisson Disaster Model Diagnosed with a CR(gamma=1) Bound
CR(gamma=1) bound
Poisson Model (RA from 5 to 6.5)
RA=6.5
RA=6.2
RA=5.9
RA=5.6
RA=5.3
RA=5
SDF mean

- The consumption growth process includes a small long-run predictable component in consumption growth and a fluctuating consumption volatility to capture economic uncertainty.

$$
\begin{aligned}
g_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
\sigma_{t+1}^2 &= \sigma^2 + \nu(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}
\end{aligned} \tag{9}
$$

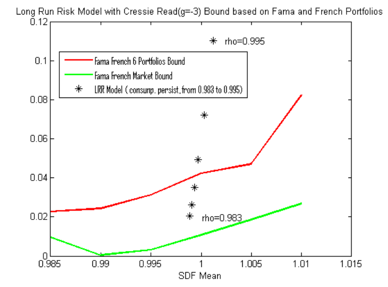
where g_t is the logarithm of real consumption growth. All innovations are $\aleph, i.i.d.(0,1)$.

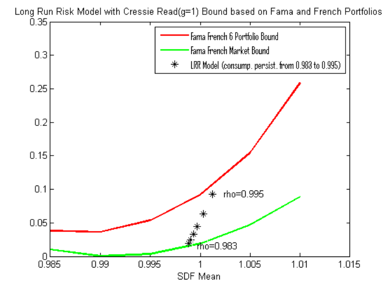
- The logarithm of the intertemporal marginal rate of substitution (IMRS) is:

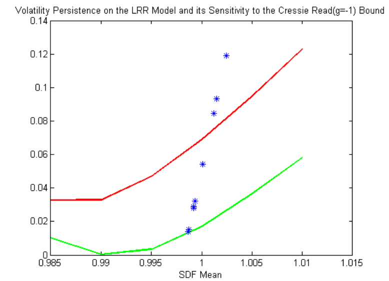
$$
m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1}
$$

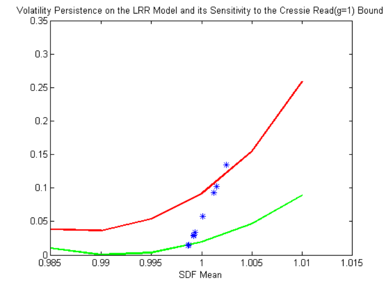
where $r_{a,t+1}$ is the return on the wealth portfolio.

Sensitivity of Entropic Discrepancies to the Persistence of Volatility and Consumption Growth in the Long Run Risk Model.









2. Fund Performance Measurement

A new class of SDF-based performance measures:

$$\alpha_i^{\gamma} = E[m(\underline{R}^{e,f}, \gamma).R_i^{e,HF}]$$

1. Empirically identified nonparametric SDFs that give more weight to returns in “bad” states of nature
2. Estimated SDFs are positive (consistent with no-arbitrage): Suitable for performance measurement.
3. Beyond mean and variance: Measures incorporate information about higher-order **mixed co-moments** $E[R_i^{e,HF}, (\lambda'.\underline{R}^{e,f})^j]_{j=1,2,...}$.
4. Flexibility: Varying specifications allow for different exposures to higher-order co-moments.

Hedge Fund Alpha as a Function of Co-moments

- Consider the implied SDF $m^{\gamma}(\underline{R}_t^{e,f}) = (1+\gamma\lambda'\underline{R}_t^{e,f})^{1/\gamma}$, with $W = \lambda'\underline{R}^{e,f}$ representing the endogenously-implied wealth return.
- Define the risk-adjusted HF return: $m.R_i^{e,HF} = f_{m,i}(W) = (1+\gamma W)^{1/\gamma} R_i^{e,HF}$
- Taylor expanding $f_{m,i}(W)$ around $E[W]$, noting that $\alpha_i = E[f_{m,i}(W)]$, and taking E(.), we obtain:

$$\begin{aligned}
\alpha_i &= (1+\gamma E(W))^{1/\gamma} E(R_i^{e,HF}) \\
&- (1+\gamma E[W])^{\frac{1-\gamma}{\gamma}} E[R_i^{e,HF}(W - E(W))] \\
&+ \frac{1}{2}(1-\gamma)(1+\gamma E[W])^{\frac{1-2\gamma}{\gamma}} E[R_i^{e,HF}(W-E(W))^2] \\
&- \frac{1}{6}(1-\gamma)(1-2\gamma)(1+\gamma E[W])^{\frac{1-3\gamma}{\gamma}} E[R_i^{e,HF}(W-E(W))^3] + \dots
\end{aligned}$$

Basis Assets

"Linear" traded portfolios:

1. CAPM: CRSP Value Weighted Market Portfolio.
2. Fama, French, and Carhart four factor model.
3. Fung and Hsieh (RFS, 2001) linear factors:
   - S&P 500, size spread (Russell 2000 - S&P 500), 10-y bond, credit spread (BAA - bond), and Emerging Market risk.

Including non-linear basis assets:

1. Fung and Hsieh (RFS, 2001) Trend following factors: PTFS factors for stocks, bonds, interest rates, FX and commodities.
2. Agarwal and Naik (RFS, 2004): ATM and OTM Put and Call portfolios.

Empirical Results: SDF Features

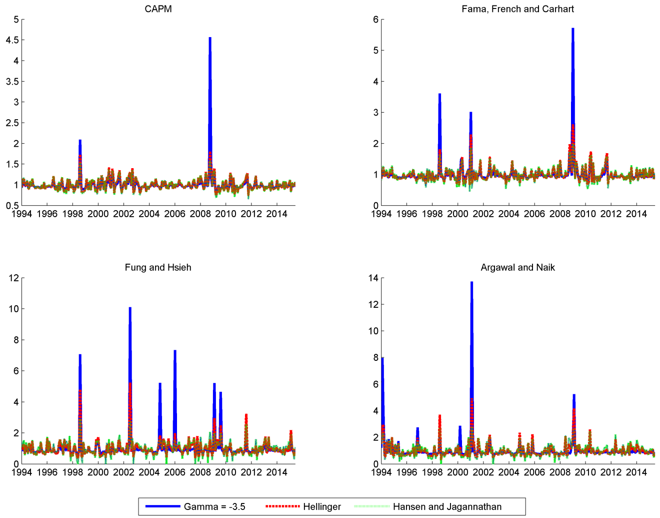
General Results:

1. Across all $\gamma's$ and basis assets pricing errors are smaller than 0.1 basis points.
2. All SDFs load positively on the market index returns (statistically significant).

Specific Results:

1. Fama, French and Carhart:
   1. High minus Low portfolio: Positive load, marginally significant.
   2. Momentum: Positive load, statistically significant.
2. Fung and Hsieh:
   1. Stock Index Lookback Straddle: Negative Load, statistically significant.
3. Agarwal and Naik: Investors sell OTM puts and hedge their position issuing ATM puts (statistically significant λs).

Implied SDF Time Series



Hedge Fund Indices - Alphas Agarwal and Naik

γ	-3.5	-2	-1	-0.5	0	0.5	1	Lin
Convertible Arbitrage	0.33	0.33	0.30	0.27	0.26	0.25	0.25	0.25
T (Boot)	(2.38)	(2.48)	(2.42)	(2.22)	(2.02)	(1.95)	(1.89)	(1.43)
CTA	0.43	0.52	0.58	0.60	0.61	0.61	0.61	0.61
T (Boot)	(2.21)	(3)	(3.85)	(4.34)	(4.43)	(4.52)	(4.38)	(4.83)
Emerging Markets	0.27	0.23	0.21	0.22	0.25	0.28	0.31	0.33
T (Boot)	(1.33)	(1.27)	(1.22)	(1.42)	(1.58)	(1.96)	(2.17)	(2.17)
Equity Market Neutral	0.38	0.36	0.34	0.33	0.33	0.33	0.33	0.33
T (Boot)	(4.45)	(4.79)	(5.53)	(5.54)	(5.75)	(5.8)	(5.92)	(5.52)
Event Driven	0.45	0.42	0.39	0.38	0.38	0.38	0.39	0.39
T (Boot)	(3.74)	(3.81)	(4.26)	(4.49)	(4.71)	(4.96)	(5.15)	(4.95)
Fixed Income Arbitrage	0.25	0.30	0.32	0.32	0.33	0.33	0.33	0.34
T (Boot)	(1.8)	(2.41)	(3)	(3.12)	(3.32)	(3.47)	(3.76)	(3.35)
Fund of Funds	0.10	0.14	0.15	0.15	0.16	0.16	0.17	0.17
T (Boot)	(1.01)	(1.54)	(1.91)	(1.98)	(2.07)	(2.3)	(2.36)	(2.36)
Global Macro	0.26	0.35	0.37	0.38	0.38	0.39	0.40	0.41
T (Boot)	(1.71)	(2.54)	(3.17)	(3.45)	(3.63)	(3.82)	(3.78)	(3.81)
Long/Short Equity Hedge	0.29	0.31	0.33	0.34	0.34	0.35	0.36	0.36
T (Boot)	(3.68)	(3.95)	(4.41)	(4.59)	(4.53)	(4.82)	(4.69)	(4.1)
Managed Futures	0.43	0.60	0.67	0.70	0.71	0.72	0.73	0.72
T (Boot)	(1.37)	(2.18)	(2.92)	(3.3)	(3.44)	(3.47)	(3.42)	(3.65)
Multi-Strategy	0.32	0.35	0.35	0.35	0.36	0.36	0.36	0.37
T (Boot)	(4.18)	(4.95)	(5.84)	(5.96)	(5.86)	(6.31)	(6.26)	(5.82)

3. Estimating Tail Risk

1. VaR / expected shortfall procedures using historical data.
2. Using options or other crash-sensitive assets.
   - Parametric assumptions about returns distributions. (Bates, 2000; Pan, 2002)
   - Short-maturity OTM options and high-frequency intraday returns to obtain jump tail risk. (Bollerslev and Todorov, 2011).
3. Making use of cross-sectional data of equity returns.
   - Common systematic tail risk factor (Kelly and Jiang, 2014)
   - Copulas: Lower tail dependence with market (Ruenzi and Weigert, 2013)

A New Tail Risk Measure

- Estimate a Tail Risk factor from a panel of equity returns.
- Euler equations allow us to extract risk-premia information embedded in those returns through a risk-neutral measure.
- Making use of Minimum Discrepancy theory we identify one RN measure, from which we obtain a Tail Risk factor.
- Mixed solution between pure historical procedures (VaR) and cross-sectional ones, but distorting probabilities with a RN measure.

From a RN measure to a Tail Risk factor

- We define our tail risk factor as the average of the risk-neutral excess expected shortfalls for each asset i at time t:

$$TR_{i,t} = E^{\mathcal{Q}(R)}[(R_{i,\tau} - VaR_{\alpha}(R_{i,\tau}))|(R_{i,\tau} \leq VaR_{\alpha}(R_{i,\tau}))] \quad (10)$$

where τ denotes the possible states of nature, α is the VaR threshold, and $\mathcal{Q}(R)$ refers to the risk neutral density

Our Tail Risk Measure

1. No need for a liquid option market and cross-section more informative than market portfolio.
    - Alternative way to Ait-Sahalia and Lo (2000) for extracting a RN measure. Generalizes to multiple assets.
2. Consistent with economic theory
    - Ait-Sahalia and Lo (2000) - Economic VAR; Giacomini and Ragusa (2013) - Forecasting with Euler conditions constraints.
    - Time-varying risk aversion - Market fears (Bollerlev & Todorov, 2011); Generalized disappointment aversion: Countercyclical risk-aversion (Routledge and Zin, 2010)
3. By choosing the time length of the window in the panel of data, we control how our measure reacts to changes in market conditions (we will calculate tail risk at a monthly frequency, thus states of nature (τ) will be daily returns).

Construction of the Tail Risk Measure

- Data: Daily returns on the 25 size and book to market Fama and French portfolios from June 1926 to April of 2014.
- Our tail risk factor is a monthly factor.
- The RNDs are estimated adopting a short panel of 30 days.
- Measure is based on 5 first principal components of the 25 portfolios (see Kosak, Nagel and Santosh, 2015).
- We estimate the RND that solves the Cressie Read problem with $\gamma = -0.5$ (Hellinger measure).

Table 1: **Principal Component Variance**

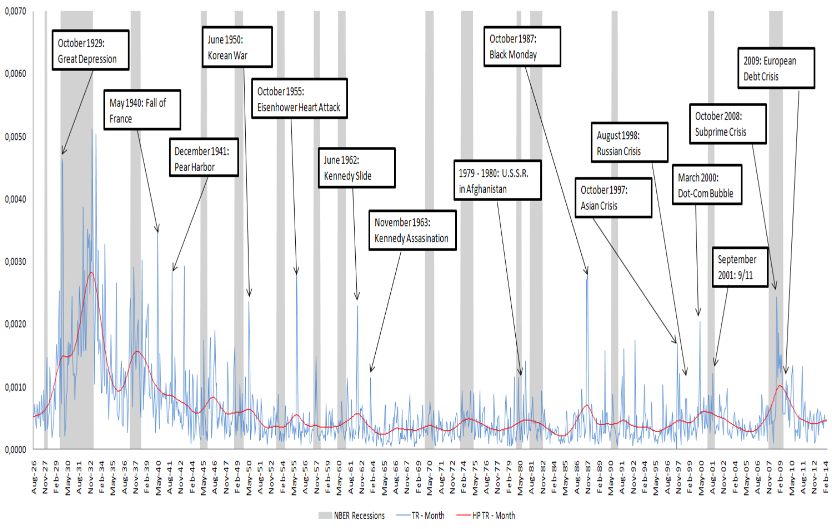
Principal Component	1	2	3	4	5
Variance (Cumulative)	0.62	0.76	0.83	0.86	0.89

This table present the first five principal components cumulative variance. Principal component analysis was performed for the hole sample for the 25 Fama and French size and book to market portfolios.

From a RN measure to a Tail Risk factor

- We compute the risk neutral expected shortfall for each principal component at the first decile of its return distribution.
- We define our tail risk factor as the average of these five shortfalls.
- The factor is high when equity returns are low.
- The risk-neutral expected shortfall corresponds up to a translation to a put price.

Time Series of Hellinger Tail risk Measure



Correlation with other tail risk measures

Option based tail risk:

1. Bollerslev, Todorov and Xu (2014).
2. VIX.

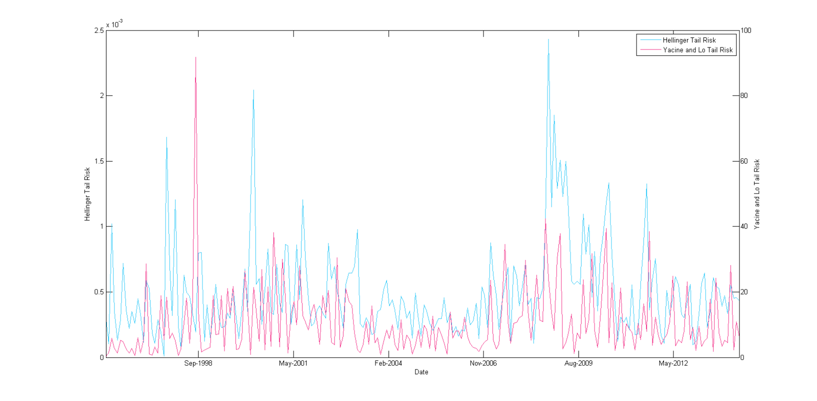
Not option based:

1. Bloom (2009)
2. Kelly and Jiang (2014).
3. Allen, Bali and Tang (2012)- CATFIN
4. Bali, Brown and Tang (2014) - Macro

Table 2: **Correlations with Other Tail Risk Measures and Financial and Macroeconomic Indicators**

	Hellinger	S&P 500	CRSP	Bloom	KJ	BTX	VIX	Macro
Hellinger	1.0000							
S&P 500	-0.3210	1.0000						
CRSP	-0.2433	0.9842	1.0000					
Bloom	0.4572	-0.1333	-0.1498	1.0000				
KJ	-0.0723	0.0854	0.0802	-0.0202	1.0000			
BTX	0.4303	-0.1293	-0.1341	0.3726	-0.2289	1.0000		
VIX	0.5581	-0.3723	-0.3709	0.9288	-0.3820	0.6625	1.0000	
Macro	0.4684	-0.0578	-0.0243	0.5809	-0.2210	0.4395	0.5548	1.0000
CATFIN	0.4507	-0.4146	-0.4385	0.3811	-0.0819	0.5206	0.6463	0.5422

Hellinger Tail risk Measure and Option-Based Tail Risk Measure



Cross-Sectional Equity Returns

1. We sort NYSE/AMEX/NASDAQ stocks into portfolios according to their sensitivity to our tail risk measure (interpreted as a hedging beta).

$$R_t^i - R_t^f = \alpha_t^i + \gamma_t^i TR_t + \epsilon_t \qquad (11)$$

2. We then track each portfolio return one month and one year ahead after the sorting.
3. We then control for various factors and compute the alphas

Sorted Portfolios - Comparison Option and Hellinger - One-month holding period - 1996 - 2014

Table 3: **Option vs. Hellinger Sorted Portfolios**

	Option Implied Tail Risk										
Portfolio	Low	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	High	High - Low
Average Return	4.63	2.13	2.48	1.26	1.78	1.17	0.85	0.97	0.81	0.90	-3.73
	(2.57)	(2.29)	(2.49)	(1.54)	(1.86)	(1.55)	(1.33)	(1.64)	(1.23)	(1.72)	(2.35)
FF3	3.51	1.47	1.74	0.50	1.15	0.55	0.31	0.40	0.36	0.56	-2.96
	(2.22)	(1.53)	(1.76)	(0.62)	(1.14)	(0.72)	(0.44)	(0.71)	(0.52)	(1.00)	(2.15)
FF3+MOM	3.72	1.32	1.54	0.47	0.93	0.48	0.30	0.45	0.31	0.64	-3.08
	(2.02)	(1.41)	(1.59)	(0.60)	(1.01)	(0.65)	(0.45)	(0.81)	(0.48)	(1.19)	(1.84)
FF3+MOM+LIQ	4.43	1.37	1.59	0.41	1.08	0.60	0.37	0.41	0.50	0.63	-3.80
	(1.72)	(1.26)	(1.42)	(0.46)	(0.91)	(0.68)	(0.49)	(0.67)	(0.62)	(1.03)	(1.60)
FF3+MOM+LIQ+VOL	-0.89	0.92	0.85	-0.19	-0.37	-0.22	0.71	0.13	0.03	1.43	2.32
	(0.33)	(0.51)	(0.49)	(0.12)	(0.25)	(0.16)	(0.59)	(0.12)	(0.03)	(1.56)	(1.02)
	Hellinger Tail Risk										
Portfolio	Low	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	High	High - Low
Average Return	4.48	2.71	2.20	1.22	1.88	0.84	0.73	0.63	0.88	1.42	-3.06
	(2.52)	(2.44)	(2.27)	(1.53)	(1.96)	(1.25)	(1.20)	(1.12)	(1.52)	(2.18)	(2.03)
FF3	3.37	1.92	1.48	0.49	1.27	0.30	0.22	0.21	0.32	0.97	-2.40
	(2.17)	(1.72)	(1.52)	(0.62)	(1.29)	(0.43)	(0.36)	(0.36)	(0.54)	(1.35)	(1.87)
FF3+MOM	3.67	1.53	1.28	0.37	1.15	0.26	0.19	0.17	0.48	1.05	-2.63
	(2.04)	(1.44)	(1.36)	(0.48)	(1.28)	(0.38)	(0.33)	(0.30)	(0.82)	(1.51)	(1.67)
FF3+MOM+LIQ	4.39	1.74	1.34	0.36	1.35	0.19	0.15	0.19	0.44	1.25	-3.14
	(1.73)	(1.34)	(1.21)	(0.42)	(1.17)	(0.25)	(0.23)	(0.29)	(0.66)	(1.58)	(1.36)
FF3+MOM+LIQ+VOL	-0.01	0.26	0.90	-0.09	0.13	0.02	-0.41	-0.02	0.15	1.48	1.49
	(0.00)	(0.15)	(0.51)	(0.07)	(0.09)	(0.01)	(0.37)	(0.02)	(0.14)	(1.16)	(0.72)

Sorted Portfolios Hellinger RN - High minus Low - 1967-2014

Table 1: **Sorted Portfolios Hellinger RN - High minus Low - 1967-2014**

	Panel A: One Month Holding Period					
Portfolio	Size-BM	Objective Shortfall	CRSP	Industry	Financial	Real
Average Return	-1.52	-0.67	-1.28	-1.44	-1.9	-1.57
	(-3.41)	(-2.50)	(-2.98)	(-3.30)	(-3.90)	(-3.73)
FF3	-1.15	-0.46	-0.98	-1.13	-1.34	-1.16
	(-3.09)	(-1.95)	(-2.59)	(-2.99)	(-3.29)	(-3.27)
FF3+MOM	-1.2	-0.46	-1.05	-1.17	-1.4	-1.21
	(-3.10)	(-1.99)	(-2.69)	(-3.02)	(-3.34)	(-3.30)
FF3+MOM+LIQ	-1.28	-0.42	-1.13	-1.22	-1.52	-1.31
	(-2.79)	(-1.81)	(-2.42)	(-2.65)	(-3.14)	(-2.99)
FF3+MOM+LIQ+VOL	-0.83	0.12	-0.77	-0.79	-0.81	-0.9
	(-1.80)	(0.46)	(-1.73)	(-1.76)	(-1.96)	(-2.19)

	Panel B: One Year Holding Period					
Portfolio	Size-BM	Objective Shortfall	CRSP	Industry	Financial	Real
Average Return	-13.84	-7.47	-9.88	-12.05	-18.18	-14.15
	(-3.10)	(-2.38)	(-2.92)	(-3.36)	(-3.91)	(-3.54)
FF3	-3.17	-1.19	-4.2	-4.28	-7.45	-4.16
	(-1.16)	(-0.45)	(-1.31)	(-1.90)	(-2.19)	(-1.72)
FF3+MOM	-4.99	-0.17	-3.21	-4.67	-9.39	-4.66
	(-2.01)	(-0.09)	(-1.32)	(-2.31)	(-3.02)	(-2.12)
FF3+MOM+LIQ	-4.98	0.79	-3.1	-5.72	-10.67	-5.61
	(-2.05)	(0.42)	(-1.16)	(-2.62)	(-3.32)	(-2.51)
FF3+MOM+LIQ+VOL	11.56	16.59	5.85	-3.15	9.87	7.62
	(1.03)	(1.75)	(1.08)	(-0.57)	(0.87)	(0.80)

1

- Given an asset pricing model $y(\theta)$, HJ (1997) measure its degree of misspecification by minimizing its quadratic distance to the set of admissible SDFs:

$$\delta_{HJ}(\theta)^2 = \min_{m \in L^2(y)} E\{(m - y(\theta))^2\} \text{ subject to } E(mx) = \pi(x) = q.$$

- HJ (1997) suggest estimating the parameter vector θ by minimizing the HJ distance:

$$\underset{\theta \in \Re^k}{argmin}\ (Exy(\theta) - q)' Exx'^{-1} (Exy(\theta) - q)$$

as an estimator alternative to the GMM (Hansen (1982)), where Exx'^{-1} is replaced by W, a symmetric positive definite matrix that usually depends on the proxy model y.

Estimating misspecified models with minimum discrepancy

- Given a proxy asset pricing model $y(\theta)$, and a convex discrepancy function ϕ, find an admissible SDF which is as close as possible to $y(\theta)$ in the ϕ discrepancy sense:

$$\delta_{MD}(\theta) = \min_{m \in L^2(y)} E\{\phi(1 + m - y(\theta))\} \text{ subject to } E(mx) = q$$

where $L^2(y) = \{m \in L^2, m >> y(\theta) - 1\}$.

- These problems should be of interest when either the asset pricing proxy model $y(\theta)$ can depend nonlinearly on the underlying primitive securities or when the underlying primitive securities themselves include assets with non-Gaussian returns.
- HJ (1997) is a particular case when $\phi(\pi) = \pi^2$.

- Let ϕ belong to the Cressie Read family: $\phi(\pi) = \frac{\pi^{\gamma+1}-1}{\gamma(\gamma+1)}$. For a fixed θ, the optimization problem specializes to:

$$\delta_{CR}(\theta) = \min_{m \in L^2(y)} E\left\{\frac{(1+m-y(\theta))^{\gamma+1}-1}{\gamma(\gamma+1)}\right\} \text{ subj. to } E(mx) = q \tag{12}$$

- For this family, Newey and Smith (2004) show that the dual belongs to the class of GEL estimators. The GEL problem dual to the MD problem is given by:

$$v_{CR}(\theta) = \max_{\lambda \in \Re^n} \lambda' q - E\left\{\frac{(\gamma\lambda' x)^{\frac{\gamma+1}{\gamma}}}{\gamma+1} + (y(\theta)-1)\lambda' x + \frac{1}{\gamma(\gamma+1)}\right\} \tag{13}$$

- The admissible SDF which is closest to the asset pricing proxy y is given by:

$$m_{CR}(\theta) = y(\theta) - 1 + (\gamma \lambda_*' x)^{\frac{1}{\gamma}}$$

where λ_* is the solution of the optimization problem (13).

- These solutions give additive correction terms to the proxy y that are nonlinear functions of the optimal linear combinations of primitive assets' payoffs $\lambda_{MD}' x$, which are the smallest corrections (in the ϕ divergence sense) for y to become an admissible SDF

- As a particular case, the HJ admissible SDF is given by a linear correction $\hat{m}_{HJ}(\theta) = y - \lambda_{HJ}' x$.

- Researchers have been using the HJ (1997) distance to estimate asset pricing models by finding the parameter vector θ^* that minimizes this distance.
- Following Kitamura (2006) and the whole literature on MD estimators, we propose estimating the above asset pricing models by finding the parameter vector θ_{MD} that minimizes any specific discrepancy function:

$$\theta_{MD} = \underset{\theta \in \Re^k}{argmin}\ \delta_{MD}(\theta),$$

- When the discrepancy belongs to the Cressie Read family:

$$\theta_{CR} = \underset{\theta \in \Re^k}{argmin}\ \underset{\lambda \in \Re^{N-1}}{max}\ \lambda' q - E\left\{ \frac{(\gamma \lambda' x)^{\frac{\gamma+1}{\gamma}}}{\gamma + 1} + (y(\theta) - 1)\lambda' x + \frac{1}{\gamma(\gamma+1)} \right\},$$

- In order to be able to perform hypothesis tests with the new proposed discrepancy measures, we develop the statistical properties of our estimators.
- We analyze the asymptotic properties of our MD estimators considering that the asset pricing models analyzed are misspecified.
- Under the same set of assumptions provided by Kitamura and Stutzer(1997) and Kitamura (2000), we prove consistency and asymptotic normality of our estimators for the family of Cressie Read discrepancies.
- We provide an application to the canonical CCAPM (estimation of the risk aversion parameter).

Conclusion

1. We propose a methodology to elicit positive admissible SDFs that price exactly a set of basis assets (risk factors).
2. We use these SDFs to construct information frontiers to evaluate the admissibility of models that imply nonlinear dynamics of fundamentals and distributions of returns with skewness and kurtosis.
3. We use these SDFs to measure performance of hedge funds.
4. We build measures of tail risk based on risk-neutralized returns of portfolios.
5. We provide estimators of the parameters of the asset pricing models that minimize the distance to admissible SDFs (a measure of misspecification).