How useful are no-arbitrage restrictions for forecasting the term structure of interest rates?

Raffaella Giacomini (with Andrea Carriero)

UCL(A) and Queen Mary

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Large literature on term structure modeling, focus on forecasting more recent
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Ang and Piazzesi (2003): affine model building on assumption of no-arbitrage in bond markets $\iff$ VAR with parameter restrictions
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- Ang and Piazzesi (2003): affine model building on assumption of no-arbitrage in bond markets ↔ VAR with parameter restrictions
  - Informal comparison of MSFE of no-arbitrage model, unrestricted VAR and random walk
Motivation

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- Ang and Piazzesi (2003): affine model building on assumption of no-arbitrage in bond markets $\iff$ VAR with parameter restrictions
  - Informal comparison of MSFE of no-arbitrage model, unrestricted VAR and random walk
- Our goal: propose a formal framework for investigating the "usefulness" of no-arbitrage restrictions for forecasting
Questions we want to answer

- Are no-arbitrage restrictions only useful because they reduce estimation error?

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No-arbitrage and forecasting

July 2008 3 / 32
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- How important is the fact that no-arbitrage restrictions incorporate a time-varying risk premium?
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- Are no-arbitrage restrictions only useful because they reduce estimation error?
- Are the results different when considering economic vs. statistical measures of accuracy?
- How important is the fact that no-arbitrage restrictions incorporate a time-varying risk premium?
- Has the usefulness of no-arbitrage restrictions changed over time?
Limitations of existing approaches

- In-sample hypothesis testing not necessarily informative

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  - Restrictions may not be true but still be useful for forecasting (e.g., bias-variance tradeoff)

Out-of-sample comparison tests (e.g., West, 1996, Clark and McCracken, 2001 etc.) not obviously applicable

We have one model and different restrictions on its parameters
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Contributions

- Propose a measure of the usefulness of parameter restrictions for forecasting and show how to perform inference.
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- The measure is tailored to the forecaster’s decision problem
- The measure can be time-varying
One model (VAR for interest rates of different maturities)
Environment

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- Atoretical restrictions (random walk)
- Statistical and economic loss functions (quadratic and utility of bond portfolio)
The idea in a nutshell

- Cast the problem in an out-of-sample forecast combination framework
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- Combine unrestricted (and random walk) forecast with no-arbitrage restricted forecast and estimate optimal weight for a general loss function
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The idea in a nutshell

- Cast the problem in an out-of-sample forecast combination framework
- Combine unrestricted (and random walk) forecast with no-arbitrage restricted forecast and estimate optimal weight for a general loss function
- Optimal weight = measure of the usefulness of no-arbitrage restrictions
- Time-varying weight = time-varying measure of usefulness
Method. Rolling forecasts

- For each yield, produce three sequences of $n$ out-of-sample forecasts using a rolling window scheme
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- unrestricted model \( \rightarrow \{ f_t^U \} \)
- model with no-arbitrage restrictions \( \rightarrow \{ f_t^R \} \)
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  - unrestricted model $\Rightarrow \{f_t^U\}$
  - model with no-arbitrage restrictions $\Rightarrow \{f_t^R\}$
  - model with random walk restrictions $\Rightarrow \{f_t^{RW}\}$

- Two forecast combinations

\[
\begin{align*}
    f_t^C &= f_t^R + (1 - \lambda) \left( f_t^U - f_t^R \right) \\
    f_t^C &= f_t^R + (1 - \lambda) \left( f_t^{RW} - f_t^R \right)
\end{align*}
\]
Method. Global measure of usefulness

For general loss $L(y_{t+1}, f_t)$:

$$\lambda^* = \arg\min_{\lambda \in R} E \left[ L \left( y_{t+1}, f_t^R + (1 - \lambda)(f_t^U - f_t^R) \right) \right]$$
Method. Global measure of usefulness

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$$\lambda^* = \arg \min_{\lambda \in R} E \left[ L \left( y_{t+1}, f_t^R + (1 - \lambda)(f_t^U - f_t^R) \right) \right]$$

Estimate out-of-sample:

$$\hat{\lambda} = \arg \min_{\lambda \in R} \frac{1}{n} \sum_{t} L \left( y_{t+1}, f_t^R + (1 - \lambda)(f_t^U - f_t^R) \right).$$
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- Large $\hat{\lambda}$ = no-arbitrage restrictions are useful
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For general loss $L(y_{t+1}, f_t)$:

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- Large $\hat{\lambda}$ = no-arbitrage restrictions are useful
- Small $\hat{\lambda}$ = no-arbitrage restrictions are not useful
Method. Local measure of usefulness

- Usefulness of economic restrictions may vary over time (similar to Giacomini and Rossi, 2008 for testing problem)
Method. Local measure of usefulness

- Usefulness of economic restrictions may vary over time (similar to Giacomini and Rossi, 2008 for testing problem)
- Local measure $\Longleftrightarrow$ time-varying weights

$$\lambda^*_t = \arg \min_{\lambda_t \in R} E \left[ L(y_{t+1}, f_t^R + (1 - \lambda_t) \left( f_U^t - f_R^t \right) \right]$$
Method. Local measure of usefulness

- Usefulness of economic restrictions may vary over time (similar to Giacomini and Rossi, 2008 for testing problem)
- Local measure $\Lam^*_t$ time-varying weights

$$
\Lam^*_t = \arg\min_{\Lam_t \in R} E \left[ L(y_{t+1}, f^R_t + (1 - \Lam_t) \left( f^U_t - f^R_t \right) \right]
$$

- Estimate a "smoothed" version of $\Lam^*_t$ over rolling windows of size $d$

$$
\hat{\Lam}_{t,d} = \arg\min_{\Lam \in R} \sum_{j=t-d+1}^{t} L(y_{t+1}, f^R_t + (1 - \Lam) \left( f^U_t - f^R_t \right) \right]
$$
Global measure $\Rightarrow$ test of $H_0 : \lambda = 0$ (restrictions are useless)

Reject if $\left| \frac{\sqrt{n\hat{\lambda}}}{\hat{\sigma}} \right| > 1.96$. We give valid $\hat{\sigma}$
Global measure $\implies$ test of $H_0: \lambda = 0$ (restrictions are useless)

Reject if $\left| \frac{\sqrt{n}\hat{\lambda}}{\hat{\sigma}} \right| > 1.96$. We give valid $\hat{\sigma}$

Local measure $\implies$ uniform confidence bands valid under $H_0: \lambda^*_t$ constant $\forall t$

$$I = (\hat{\lambda}_{t,d} - k_{\alpha,\pi} \frac{\hat{\sigma}}{\sqrt{d}}, \hat{\lambda}_{t,d} + k_{\alpha,\pi} \frac{\hat{\sigma}}{\sqrt{d}})$$

We give $k_{\alpha,\pi}$ (same as Giacomini and Rossi, 2008)
Method. Inference

- Global measure $\implies$ test of $H_0 : \lambda = 0$ (restrictions are useless)

  $\text{Reject if } \left| \frac{\sqrt{n} \hat{\lambda}}{\hat{\sigma}} \right| > 1.96. \text{ We give valid } \hat{\sigma}$

- Local measure $\implies$ uniform confidence bands valid under $H_0 : \lambda_t^* \text{ constant } \forall t$

  $$I = \left( \hat{\lambda}_{t,d} - k_{\alpha,\pi} \frac{\hat{\sigma}}{\sqrt{d}}, \hat{\lambda}_{t,d} + k_{\alpha,\pi} \frac{\hat{\sigma}}{\sqrt{d}} \right)$$

  We give $k_{\alpha,\pi}$ (same as Giacomini and Rossi, 2008)

- If $0 \not\in I$ at some $t$ $\implies$ reject that restrictions were useless $\forall t$
For a quadratic loss $L(y_{t+1}, f_t) = (y_{t+1} - f_t)^2$ measure of usefulness is

$$\lambda^* = \frac{E \left[ (y_{t+1} - f_t^U) (f_t^R - f_t^U) \right]}{E \left[ (f_t^R - f_t^U)^2 \right]}$$
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Estimate by OLS

$$y_{t+1} - f_t^U = \lambda (f_t^R - f_t^U) + \text{error}$$
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Local measure obtained by estimating rolling regressions
Statistical loss

- For a quadratic loss $L(y_{t+1}, f_t) = (y_{t+1} - f_t)^2$ measure of usefulness is

$$\lambda^* = \frac{E\left[(y_{t+1} - f_t^U) (f_t^R - f_t^U)\right]}{E\left[(f_t^R - f_t^U)^2\right]}$$

- Estimate by OLS

$$y_{t+1} - f_t^U = \lambda (f_t^R - f_t^U) + \text{error}$$

- Local measure obtained by estimating rolling regressions
- $\hat{\sigma}$ for the test and confidence bands is OLS standard error (HAC)
Illustrative example

Suppose $y_t = \beta' x_t + \varepsilon_t$, $x_t \sim iid(0, \sigma^2_x I_k)$, $\varepsilon_t \sim iid(0, \sigma^2_\varepsilon)$
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Forecasts: $f^U_t = \hat{\beta} x_{t+1}$ and $f^R_t = \tilde{\beta} x_{t+1}$. If $\hat{\beta}$ and $\tilde{\beta}$ independent

$$\lambda^* = \frac{tr(Var(\hat{\beta}))}{tr \left( Var(\hat{\beta}) + Var(\tilde{\beta}) + \left( bias(\tilde{\beta}) \right)^2 \right)}$$
Suppose \( y_t = \beta' x_t + \varepsilon_t, \ x_t \sim iid(0, \sigma^2 x I_k), \ \varepsilon_t \sim iid(0, \sigma^2 \varepsilon) \)

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\[
\lambda^* = \frac{tr(\text{Var}(\hat{\beta}))}{tr\left( \text{Var}(\hat{\beta}) + \text{Var}(\tilde{\beta}) + (\text{bias}(\tilde{\beta}))^2 \right)}
\]

\( \lambda^* \approx 0 \) (restrictions not useful) if large bias of restricted estimator (\( \Leftrightarrow \) restrictions too misspecified to be useful).
Suppose \( y_t = \beta' x_t + \varepsilon_t, \quad x_t \sim iid(0, \sigma_x^2 I_k), \quad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \)

Forecasts: \( f^U_t = \tilde{\beta} x_{t+1} \) and \( f^R_t = \tilde{\beta} x_{t+1} \). If \( \tilde{\beta} \) and \( \tilde{\beta} \) independent

\[
\lambda^* = \frac{tr(Var(\tilde{\beta}))}{tr(Var(\tilde{\beta}) + Var(\tilde{\beta}) + (bias(\tilde{\beta}))^2)}
\]

\( \lambda^* \approx 0 \) (restrictions not useful) if large bias of restricted estimator (\( \iff \) restrictions too misspecified to be useful).

\( \lambda^* \approx 1 \) (restrictions useful) if large variance of unrestricted estimator (\( \iff \) usefulness due to reduction in estimation uncertainty)
Limitations of statistical measures of accuracy

- Usefulness of no-arbitrage restrictions may be simply due to a reduction in dimensionality
Limitations of statistical measures of accuracy

- Usefulness of no-arbitrage restrictions may be simply due to a reduction in dimensionality
- Generally true for out-of-sample comparisons based on MSE
Economic loss function. Portfolio utility loss

- Similar to West, Edison and Cho (1993)
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Bond portfolio $w'y_{t+1}$. Optimal weights for quadratic utility

$$w^* = \arg \min_w \left\{ w' E_t [y_{t+1}] - \frac{\gamma}{2} w' \Sigma w \right\}$$

$$\Sigma = \text{Var}(y_{t+1}) \text{ and } \gamma = \text{CRRA}/ (\text{CRRA} + 1)$$
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$$\Sigma = \text{Var}(y_{t+1}) \text{ and } \gamma = \frac{\text{CRRA}}{\gamma + 1}$$

- Classical solution (Markowitz, 1952)

$$w^* = C_1 + C_2 E_t [y_{t+1}],$$
$$C_1 = \frac{\Sigma^{-1} l}{l' \Sigma^{-1} l}, \quad C_2 = \frac{1}{\gamma} \left( \Sigma^{-1} - \frac{\Sigma^{-1} u' \Sigma^{-1}}{l' \Sigma^{-1} l} \right)$$
Economic loss function. Portfolio utility loss

- Different conditional mean forecasts ⇔ different weights
Economic loss function. Portfolio utility loss

- Different conditional mean forecasts $\Leftrightarrow$ different weights
- For the combination forecast

$$w^*(\lambda) = C_1 + C_2 \left( f^R_t + (1 - \lambda)(f^U_t - f^R_t) \right)$$

Estimate by substituting $\mathbb{E}$ with out-of-sample mean (global) or rolling out-of-sample means (local)

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No-arbitrage and forecasting
Economic loss function. Portfolio utility loss

- Different conditional mean forecasts $\iff$ different weights
- For the combination forecast

$$w^*(\lambda) = C_1 + C_2 \left( f_t^R + (1 - \lambda)(f_t^U - f_t^R) \right)$$

- Measure of usefulness

$$\lambda^* = \arg \min_{\lambda \in R} \left[ -w^*(\lambda)' E[y_{t+1}] + \frac{\gamma}{2} w^*(\lambda)' \Sigma w^*(\lambda) \right]$$

$$= \frac{E \left[ (f_t^R - f_t^U)' C_2 (y_{t+1} - \gamma \Sigma (C_1 + C_2 f_t^U)) \right]}{E \left[ \gamma (f_t^R - f_t^U)' C_2 \Sigma C_2 (f_t^R - f_t^U) \right]}$$
Economic loss function. Portfolio utility loss

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- Measure of usefulness

\[ \lambda^* = \arg\min_{\lambda \in \mathcal{R}} \left[ -w^*(\lambda)' E[y_{t+1}] + \frac{\gamma}{2} w^*(\lambda)' \Sigma w^*(\lambda) \right] = \frac{E \left[ (f_t^R - f_t^U)' C_2' (y_{t+1} - \gamma \Sigma (C_1 + C_2 f_t^U)) \right]}{E \left[ \gamma (f_t^R - f_t^U)' C_2' \Sigma C_2 (f_t^R - f_t^U) \right]} \]

- Estimate by substituting \( E[\cdot] \) with out-of-sample mean (global) or rolling out-of-sample means (local)
Empirical application. Ang and Piazzesi’s model

- \( y_t = \) vector of yields of different maturities (1, 3, 12, 36, 60 months)
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- $y_t =$ vector of yields of different maturities (1, 3, 12, 36, 60 months)
- State-space representation:
  
  $y_t = A + BF_t + \nu_t$
  
  $F_t = \Psi F_{t-1} + \Omega \epsilon_t$

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Empirical application. Ang and Piazzesi’s model

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- State-space representation:

  $$y_t = A + BF_t + \nu_t$$
  $$F_t = \Psi F_{t-1} + \Omega \varepsilon_t$$

- $F_t$ contains three latent factors ($\approx$ level, slope and curvature of term structure)
Empirical application. AP model as a restricted VAR

- No-arbitrage assumption restricts the elements of $A$ and $B$

\[ \bar{A}_{n+1} = \bar{A}_n + \bar{B}'_n(-\Lambda_0) + 0.5\bar{B}'_n\Omega\Omega'\bar{B}_n - \delta_0 \]

\[ \bar{B}'_{n+1} = \bar{B}'_n(\Psi - \Omega\Lambda_1) - \delta'_1 \]

with $A_n = -\bar{A}_n/n$, $B_n = -\bar{B}_n/n$, $\bar{A}_1 = -\delta_0$ and $\bar{B}_1 = -\delta_1$
Empirical application. AP model as a restricted VAR

- No-arbitrage assumption restricts the elements of $A$ and $B$
  \[
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  \tilde{B}'_{n+1} = \tilde{B}'_n(\Psi - \Omega\Lambda_1) - \delta'_1
  \]
  with $A_n = -\tilde{A}_n/n$, $B_n = -\tilde{B}_n/n$, $\tilde{A}_1 = -\delta_0$ and $\tilde{B}_1 = -\delta_1$

- $\Lambda_0$ and $\Lambda_1$ are such that $\Lambda_t = \Lambda_0 + \Lambda_1 F_t$, with $\Lambda_t = \text{market prices of risk}$

\[
\Lambda_1 \neq 0 \implies \text{time-varying risk premium} \\
\Lambda_1 = 0 \implies \text{constant risk premium}
\]
Empirical application. AP model as a restricted VAR

- AP state-space model \( \approx MA(\infty) \implies \text{approximate with VAR}(3) \)

\[
Y = X\Phi + U
\]
Empirical application. AP model as a restricted VAR

- AP state-space model $\approx MA(\infty) \implies$ approximate with VAR(3)
  
  $$Y = X\Phi + U$$

- Impose the restrictions on the VAR by writing the sample likelihood of $Y$ as
  
  $$|\Sigma_u|^{-T/2} \exp\left\{-0.5 \text{tr}[\Sigma_u^{-1} (\Gamma_{yy}^* - \Phi'\Gamma_{xy}^* - \Gamma_{xy}^*\Phi + \Phi'\Gamma_{xx}^*\Phi)]\right\}$$

  $\Gamma_{yy}^*, \Gamma_{yx}^*, \Gamma_{xx}^* =$ moments implied by state-space
Empirical application. AP model as a restricted VAR

- AP state-space model \( \approx MA(\infty) \iff \) approximate with VAR(3)
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  \]

  \( \Gamma_{yy}, \Gamma_{yx}, \Gamma_{xx} = \) moments implied by state-space

- Estimate by ML \( \iff \) no-arbitrage restricted estimator

Carriero and Giacomini (QM and UCL(A)) (UCL(A) and Queen Mary)
Empirical application. AP model as a restricted VAR

- AP state-space model $\approx MA(\infty) \implies$ approximate with VAR(3)

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$\Gamma_{yy}^*, \Gamma_{yx}^*, \Gamma_{xx}^* = $ moments implied by state-space

- Estimate by ML $\implies$ no-arbitrage restricted estimator

- The unrestricted estimator is the OLS estimator
Empirical results

VERY PRELIMINARY....
Quadratic loss. Global measure of usefulness

### NA vs. unrestricted VAR

<table>
<thead>
<tr>
<th>Yields</th>
<th>$\hat{\lambda}$</th>
<th>Test of $H_0$ : NA restrictions useless</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-month</td>
<td>0.427</td>
<td>4.711*</td>
</tr>
<tr>
<td>3-month</td>
<td>0.655</td>
<td>3.547*</td>
</tr>
<tr>
<td>12-month</td>
<td>0.979</td>
<td>5.374*</td>
</tr>
<tr>
<td>36-month</td>
<td>0.814</td>
<td>4.823*</td>
</tr>
<tr>
<td>60-month</td>
<td>0.826</td>
<td>5.424*</td>
</tr>
</tbody>
</table>

Carriero and Giacomini (QM and UCL(A))

No-arbitrage and forecasting

July 2008
### NA with constant risk premium vs. unrestricted VAR

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<tbody>
<tr>
<td>1-month</td>
<td>0.337</td>
<td>3.596*</td>
</tr>
<tr>
<td>3-month</td>
<td>0.785</td>
<td>4.559*</td>
</tr>
<tr>
<td>12-month</td>
<td>0.796</td>
<td>4.777*</td>
</tr>
<tr>
<td>36-month</td>
<td>0.912</td>
<td>5.894*</td>
</tr>
<tr>
<td>60-month</td>
<td>0.742</td>
<td>5.251*</td>
</tr>
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Quadratic loss. Global measure of usefulness

**NA vs. random walk**

<table>
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<tr>
<th>Yields</th>
<th>$\hat{\lambda}$</th>
<th>Test of $H_0$ : NA restrictions useless</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-month</td>
<td>0.516</td>
<td>4.649*</td>
</tr>
<tr>
<td>3-month</td>
<td>-0.110</td>
<td>-0.526</td>
</tr>
<tr>
<td>12-month</td>
<td>0.276</td>
<td>0.828</td>
</tr>
<tr>
<td>36-month</td>
<td>0.099</td>
<td>0.337</td>
</tr>
<tr>
<td>60-month</td>
<td>0.274</td>
<td>0.668</td>
</tr>
</tbody>
</table>
NA vs. unrestricted VAR
NA with constant risk premium vs. unrestricted VAR
Quadratic loss. Local measure of usefulness

NA vs. random walk

![Graphs showing NA vs. random walk for different time periods](image-url)
### Results for CRRA = 1

<table>
<thead>
<tr>
<th>Comparison</th>
<th>$\lambda$</th>
<th>$H_0$: NA restr. useless</th>
</tr>
</thead>
<tbody>
<tr>
<td>NA vs. unrestricted VAR</td>
<td>0.416</td>
<td>23.33*</td>
</tr>
<tr>
<td>NA with const. risk premium vs. unrestricted VAR</td>
<td>0.199</td>
<td>26.80*</td>
</tr>
<tr>
<td>NA vs. random walk</td>
<td>0.512</td>
<td>63.25*</td>
</tr>
</tbody>
</table>
Portfolio utility loss. Local measure of usefulness

NA vs. unrestricted VAR

Graph showing the comparison between NA and unrestricted VAR over the years from 1984 to 2002.
NA with constant risk premium vs. unrestricted VAR
Portfolio utility loss. Local measure of usefulness

NA vs. random walk

![Graph showing NA vs. random walk](image-url)
Conclusion. Theoretical framework

- No-arbitrage affine term structure model = VAR with theory-based parameter restrictions
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- No-arbitrage affine term structure model = VAR with theory-based parameter restrictions
- Proposed a framework for measuring and performing inference about the usefulness of restrictions for forecasting
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- Usefulness depends on the forecaster’s loss function
No-arbitrage affine term structure model = VAR with theory-based parameter restrictions

Proposed a framework for measuring and performing inference about the usefulness of restrictions for forecasting

Usefulness depends on the forecaster’s loss function

Considered both global and local measures
Conclusion. Back to our initial questions

- Are no-arbitrage restrictions only useful because they reduce estimation error?

Yes. In terms of statistical accuracy, they are no better than atheoretical ways to do the same.

Yes. Dimension reduction not so important economically.

Not important for statistical accuracy, but essential for constructing bond portfolios.

Yes, they have become less useful.
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No-arbitrage and forecasting

July 2008
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