Loss-Deviation risk measures

Abstract

In this paper we present a class of risk measures composed of coherent risk measures with generalized deviation measures. Based on the Limitedness axiom, we prove that this set is a sub-class of coherent risk measures. We present extensions of this result for the case of convex or co-monotone coherent risk measures. Under this perspective, we propose a specific formulation that generates, from any coherent measure $\rho$, a generalized deviation based on the dispersion of results worse than $\rho$, which leads to a very interesting risk measure. Moreover, we present some examples of risk measures that lie in our proposed class.

Keywords: Coherent risk measures, Generalized deviation measures, Convex risk measures, Co-monotone coherent risk measures, Limitedness.

1 Introduction

The definition of risk is based on two main concepts: the possibility of a negative outcome, i.e., a loss, and the variability in terms of an expected result, i.e., a deviation. Since the time at which the modern theory of finance was accepted, the role of risk measurement has attracted attention. Initially, it was predominantly used as a dispersion measure, such as variance, which contemplates the second pillar of the definition. More recently, the occurrence of critical events has turned the attention to tail risk measurement, as is the case with the well-known Value at Risk (VaR) and Expected Shortfall (ES) measures, which contemplate the first pillar of the definition. Moreover, theoretical and mathematical discussions have gained attention in the literature, giving importance to distinct axiomatic structures for classes of risk measures and their properties.

Despite their fundamental importance, such classes present a very wide range for those risk measures that can be understood as valid or useful. Thus, they can be considered as a first step, in which measures with poor theoretical
properties are discarded. The next step would be to consider, inside a class, those measures more suited to practical use. Thus, to ensure a more complete measurement it is reasonable to consider contemplating both pillars of risk definition, which are the possibility of negative results and variability over an expected result, as a single measure.

Some authors have proposed and studied specific examples of risk measures of this kind. Ogryczak and Ruszczyński (1999) analyzed properties from the mean plus semi-deviation. Fischer (2003) and Chen and Wang (2008) considered combining the mean and semi-deviations at different powers to form a coherent risk measure. Furman and Landsman (2006) proposed a measure that weighs the mean and standard deviation in the truncated tail by VaR. Krokhmal (2007) extended the ES concept, obtained as the solution to an optimization problem, for cases with higher moments with a relationship including deviation measures. Righi and Ceretta (2015) considered penalizing the ES by the dispersion of results that represent losses exceeding the ES.

These risk measures are individual examples, rather than an entire class. The difficulty in combining both concepts arises from the loss of theoretical properties of individual components, especially the fundamental Monotonicity axiom. This property guarantees that positions that lead to the worst outcomes have larger values for risk measures. For instance, the axiom does not have a very intuitive mean plus standard deviation measure, despite the very good characteristics and intuitive separate meaning of both the mean and standard deviation.

Seeking to address this deficiency, our objective in this paper is to present a whole class of risk measures of the form $\rho + D$. In our main context, $\rho$ is a coherent risk measure in the sense of Artzner et al. (1999), whereas $D$ is a generalized deviation measure, as proposed by Rockafellar et al. (2006). We prove a simple but very useful result that relates Limitedness, an axiom we propose of the form $\rho(X) \leq -\inf(X) = \sup -X$, with Monotonicity and Lower Range Dominance. Thus, we can state that this set of measures is a sub-class of coherent risk measures. Moreover, we also discuss issues regarding Law Invariance and representations introduced in Kusuoka (2001). Our results can be extended to the case of convex risk measures in the sense of Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002), or co-monotone coherent risk measures, as for the spectral class proposed by Acerbi (2002) or distortion risk measures used in insurance. Under this perspective, we expose a formulation that generates, from any coherent measure $\rho$, a generalized deviation $D$ based on the dispersion of results worse than $\rho$. Thus, combining both the selected $\rho$ and generated $D$ produces a risk measure that lies on our proposed class. This kind of risk measure serves as a more solid
protection, once it yields higher values due to the penalty resulting from dispersion. This formulation can be extended to the case of convex risk measures. We also present specific examples of risk measures that lie on the proposed class to illustrate our theoretical results.

Our results contribute to existing knowledge in the literature because, to the best of our knowledge, no whole class or sub-class of risk measures, such as that proposed by us, has been considered in previous studies. Rockafellar et al. (2006) presented an interplay between coherent risk measures and generalized deviation measures, and Rockafellar and Uryasev (2013) proposed a risk quadrangle, where this relationship is extended by adding intersections with concepts of error and regret under a generator statistic. However, these studies are centered on an interplay of concepts, rather than a class of measures that join both pillars of the definition of risk. Filipović and Kupper (2007) presented results in which convex functions possess Monotonicity and Translation Invariance, both of which are convex risk measures. Nonetheless, their result is based on the supremum of functions on a vector space, and not on a relation of axioms for a class of risk measures such as in our approach.

Our results also contribute to the financial industry because the formulation based on our proposed class possesses solid theoretical properties that ensure its use without violating axiomatic assumptions. The most evident application is practical risk measurement because these kinds of measures consider the two main pillars of the risk concept, beyond which they are coherent. Another application is in the determination of capital requirement. As there is a penalization that represents the dispersion of negative results, greater protection can be achieved, which leads to a lower chance of default. Our findings can also be applied to resource allocation. A fundamental aspect of portfolio strategies is optimization of a convex function. This is of course the case with our risk measures. Moreover, their functional form can be interesting in the same way as the mean plus standard deviation for Markowitz-based pricing models.

The remainder of this paper is structured as follows: section 2 presents the notation, definitions as well as preliminaries from results in the literature; section 3 contains the main results relating to our class; section 4 presents the formulation we propose for generating risk measures, besides some specific examples; section 5 concludes the paper.

2 Preliminaries

In this section we present the notation, definitions, and previous results from the literature that are used throughout the paper. Unless otherwise stated,
the content is based on the following notation. Consider the random result $X$ of any asset ($X \geq 0$ is a gain, $X < 0$ is a loss) that is defined in an atom-less probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is the sample space, $\mathcal{F}$ is the set of possible events in $\Omega$, and $\mathbb{P}$ is a probability measure that is defined in $\Omega$ of the events contained in $\mathcal{F}$. Thus, $E_\mathbb{P}[X]$ is the expected value of $X$ under $\mathbb{P}$. $\sigma_\mathbb{P}(X)$ and $\sigma^{-}_\mathbb{P}(X)$ are the regular and the semi-standard deviation of $X$ under $\mathbb{P}$, respectively. $\text{corr}_\mathbb{P}(X,Y)$ is the correlation between $X$ and $Y$ under $\mathbb{P}$. In addition, $\mathcal{P} = \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P}\}$ is a nonempty set, because $\mathbb{P} \in \mathcal{P}$, which represents the measures $\mathbb{Q}$ defined in $\Omega$, are absolutely continuous in relation to $\mathbb{P}$. $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the density of $\mathbb{Q}$ relative to $\mathbb{P}$, which is known as the Radon-Nikodym derivative. $\mathcal{P}_{[0,1]}$ is the set of probability measures defined in $[0,1]$. All equalities and inequalities are considered to be almost surely in $\mathbb{P}$. $F_X$ is the probability function of $X$ and its inverse is $F_X^{-1}$. Because $(\Omega, \mathcal{F}, \mathbb{P})$ is atom-less, $F_X$ can be assumed to be continuous. Let $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ be the space of random variables of which $X$ is an element, with $1 \leq p \leq \infty$, as defined by the norm $\|X\|_p = (E_\mathbb{P}[|X|^p])^{\frac{1}{p}}$ with finite $p$ and $\|X\|_\infty = \inf \{k : |X| \leq k\}$. $X \in L^p$ indicates that $\|X\|_p < \infty$, implying that the absolute value of $X$ to the $p$ power is limited and integrable. We have that $L^q, \frac{1}{p} + \frac{1}{q} = 1$, is the dual space of $L^p$. Further, $(X)^- = \max(-X,0)$. In this context, measuring risk is equivalent to establishing the function $\rho : L^p \rightarrow \mathbb{R}$; in other words, summarizing the risk of position $X$ into one number.

We begin by defining the theoretical properties that are axioms for risk measures. There is an extremely large number of possible properties because risk measures are functions. We focus on those that are most prominent in the literature and that are used in this paper. Each class of risk measures is based on a specific set of axioms. We also define the classes of risk measures that are representative in this paper.

**Definition 2.1.** Let $\rho : L^p \rightarrow \mathbb{R}$. $\rho$ may fulfills the following properties:

(i) **Monotonicity:** if $X \leq Y$, then $\rho(X) \geq \rho(Y), \forall X, Y \in L^p$.

(ii) **Translation Invariance:** $\rho(X + C) = \rho(X) - C, \forall X \in L^p, C \in \mathbb{R}$.

(iii) **Sub-additivity:** $\rho(X + Y) \leq \rho(X) + \rho(Y), \forall X, Y \in L^p$.

(iv) **Positive Homogeneity:** $\rho(\lambda X) = \lambda \rho(X), \forall X \in L^p, \lambda \geq 0$

(v) **Convexity:** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y), \forall X, Y \in L^p, 0 \leq \lambda \leq 1$.

(vi) **Co-monotonic Additivity:** $\rho(X + Y) = \rho(X) + \rho(Y), \forall X, Y \in L^p$ with $X, Y$ co-monotone.
(vii) Translation Insensitivity: $\rho(X + C) = \rho(X), \forall X \in L^p, C \in \mathbb{R}$

(viii) Non-negativity: $\rho(X) \geq 0, \forall X \in L^p$.

(ix) Law Invariance: if $F_X = F_Y$, then $\rho(X) = \rho(Y), \forall X, Y \in L^p$.

(x) Lower Range Dominance: $\rho(X) \leq E[|X|] - \inf X, \forall X \in L^p$.

(xi) Limitedness: $\rho(x) \leq -\inf X = \sup -X, \forall X \in L^p$.

(xii) Fatou Continuity: if $|X_n| \leq Y, \{X_n\}_{n=1}^\infty, Y \in L^p$, and $|X_n| \to X$, then $\rho(X) \leq \lim inf \rho(|X_n|)$.

Remark 2.2. Monotonicity requires that if one position generates worse results than another, then its risk shall be greater. Translation Invariance ensures that if a certain gain is added to a position, its risk shall decrease by the same amount. These two axioms together imply Lipschitz continuity as $|\rho(X) - \rho(Y)| \leq \|X - Y\|_p$. Sub-additivity, which is based on the principle of diversification, implies that the risk of a combined position is less than the sum of the individual risks. Positive Homogeneity is related to the position size, i.e., the risk proportionally increases with position size. These two axioms together are known as sub-linearity. Convexity is a well-known property of functions that can be understood as a relaxed version of sub-linearity. In the presence of any two axioms among Sub-additivity, Convexity, and Positive Homogeneity, the third is the consequence. Co-monotonic Additivity is an extreme case where there is no diversification, because the positions have perfect positive association. Co-monotonic Additivity implies Positive Homogeneity. Translation Insensitivity indicates that the risk in relation to the expected value does not change if a constant value is added. Non-negativity guarantees that any position exhibits non-negative risk. Law invariance ensures that two positions with the same probability function have equal risks. Lower Range Dominance restricts the measure to a range that is lower than the range between the expected value and the minimum value. Limitedness ensures that the risk of a position is never greater than the maximum loss. The Fatou continuity is a well-established property for functions, directly linked to lower semi-continuity and continuity from above.

Definition 2.3. Let $\rho : L^p \to \mathbb{R}$ and $D : L^p \to \mathbb{R}_+$.

(i) $\rho$ is a coherent risk measure in the sense of Artzner et al. (1999) if it fulfills the axioms of Monotonicity, Translation Invariance, Sub-additivity, and Positive Homogeneity.
(ii) \( \rho \) is a convex risk measure in the sense of Föllmer and Schied (2002) Frittelli and Rosazza Gianin (2002) if it fulfills the axioms of Monotonicity, Translation Invariance, and Convexity.

(iii) \( D \) is a generalized deviation measure in the sense of Rockafellar et al. (2006) if it fulfills the axioms of Translation Insensitivity, Non-negativity, Sub-additivity, and Positive Homogeneity.

(iv) \( D \) is a convex deviation measure in the sense of Pflug (2006) if it fulfills the axioms of Translation Insensitivity, Non-negativity, and Convexity.

(v) A risk measure is said to be law invariant, lower range dominated, limited, co-monotone, or Fatou continuous if it fulfills the axioms of Law Invariance, Lower Range Dominance, Limitedness, Co-monotonic Additivity, or Fatou Continuity, respectively.

Remark 2.4. Given a coherent risk measure \( \rho \), it is possible to define an acceptance set as \( A_\rho = \{ X \in L^p : \rho(X) \leq 0 \} \) of positions that cause no loss. Let \( L^p_+ \) be the cone of the non-negative elements of \( L^p \) and \( L^p_- \) its negative counterpart. This acceptance set contains \( L^p_+ \), has no intersection with \( L^p_- \), and is a convex cone. The risk measure associated with this set is \( \rho(X) = \inf \{ m : X + m \in A_\rho \} \), i.e., the minimum capital that needs to be added to \( X \) to ensure it becomes acceptable. For convex risk measures, \( A_\rho \) need not be a cone.

A coherent risk measure can be represented as the worst possible expectation from scenarios generated by probability measures \( Q \in \mathcal{P} \), known as dual sets. Artzner et al. (1999) presented this result for discrete \( L^\infty \) spaces. Delbaen (2002) generalized the result for continuous \( L^\infty \) spaces, whereas Inoue (2003) considered the spaces \( L^p, 1 \leq p \leq \infty \). Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) presented a similar result for convex risk measures based on a penalty function. It is also possible to represent generalized deviation measures in a similar approach, with the due adjustments, as demonstrated by Rockafellar et al. (2006). Pflug (2006) proved similar results for convex deviation measures also based on a penalty function. In this sense, the dual representations we consider in this paper are formally guaranteed by the following results.

Theorem 2.5. Let \( \rho : L^p \to \mathbb{R} \) and \( D : L^p \to \mathbb{R}_+ \). Then:

(i) \( \rho \) is a Fatou continuous coherent risk measure if, and only if, it can be represented as \( \rho(X) = \sup_{Q \in \mathcal{P}_\rho} E_Q[-X], \) where \( \mathcal{P}_\rho \subseteq \mathcal{P}_q = \{ Q \in \mathcal{P} : \frac{dQ}{dP} \in L^q, \frac{dQ}{dP} \geq 0, E_P[\frac{dQ}{dP}] = 1 \} \) is a closed and convex dual set.
(ii) $\rho$ is a Fatou continuous convex risk measure if, and only if, it can be represented as $\rho(X) = \sup_{Q \in P_\rho} \{E_Q[-X] - \gamma_\rho(Q)\}$, where $\gamma_\rho : L^\gamma \to \mathbb{R} \cup \{\infty\}$ is a lower semi-continuous convex penalty function conform $\gamma_\rho(Q) = \sup_{X \in A_\rho} E_Q[-X]$, with $\gamma_\rho(Q) \geq -\rho(0)$.

(iii) $D$ is a Fatou continuous generalized deviation measure if, and only if, it can be represented as $D(X) = E_P[X] - \inf_{Q \in P_D} E_Q[X]$, where $P_D = \{Q \in P : \frac{dQ}{dP} \in L^\gamma, E_P[\frac{dQ}{dP}] = 1, D(X) \geq E_P[X] - E_Q[X], \forall X \in L^p\}$ is a closed and convex dual set. Moreover, $D$ is lower range dominated if and only if $\frac{dQ}{dP} \geq 0, \forall Q \in P_D$.

(iv) $D$ is a Fatou continuous convex deviation measure if, and only if, it can be represented as $D(X) = E_P[X] - \inf_{Q \in P_D} \{E_Q[X] + \gamma_D(Q)\}$, where $\gamma_D$ is similar to $\gamma_\rho$. Moreover, $D$ is lower range dominated if and only if $\frac{dQ}{dP} \geq 0, \forall Q \in P_D$.

3 Main Results

This section contains our main contribution. We consider limited risk measures of the form $\rho + D$, with $\rho$ a coherent risk measure, and $D$ a generalized deviation measure. Note that if $D$ is a generalized deviation measure, then so is $\beta D$ for $\beta \geq 0$. We claim that this kind of measure is a sub-class of coherent risk measures, with proper dual representation. In that regard, we initially prove simple but very interesting results that relate Monotonicity and Lower Range Dominance axioms to Limitedness. Based on these results, and those from section 2, we are able to prove our main theorem. Our results can be extended to the convex and co-monotone coherent cases.

**Proposition 3.1.** Let $\rho : L^p \to \mathbb{R}$ and $D : L^p \to \mathbb{R}_+$. Then:

(i) If $\rho$ fulfills Sub-additivity or Convexity, and Limitedness, then it possesses Monotonicity.

(ii) If $\rho$ fulfills Translation Invariance and Monotonicity, then it possesses Limitedness.

(iii) If $\rho$ is a coherent (convex) risk measure, then it fulfills Limitedness.

(iv) If $\rho + D$ is a coherent (convex) risk measure, then $D$ possesses Lower Range Dominance.
Proof. For (i), we begin by supposing the Sub-additivity of \( \rho \). Let \( X, Y \in L^p, X \leq Y \). There is \( Z \in L^p, Z \geq 0 \) such that \( Y = X + Z \). By Limitedness we must have \( \rho(Z) \leq -\inf Z \leq 0 \). Thus, by Sub-additivity we obtain \( \rho(Y) = \rho(X + Z) \leq \rho(X) + \rho(Z) \leq \rho(X) \), as required. By the same logic, let \( \rho \) have Convexity. Thus, for \( 0 \leq \lambda \leq 1 \) we have \( Y = \lambda X + (1 - \lambda)Z \). This leads to \( \rho(Y) = \rho(\lambda X + (1 - \lambda)Z) \leq \lambda \rho(X) + (1 - \lambda)\rho(Z) \leq \lambda \rho(X) \). As \( \lambda \) is an arbitrary value in \([0, 1]\), we obtain \( \rho(Y) \leq \rho(X) \), as desired.

For (ii), note that because \( X \geq \inf X \), Monotonicity and Translation Invariance implies \( \rho(X) \leq \rho(\inf X) = -\inf X \), which is Limitedness.

We have that (iii) is directly implied by (ii), because a coherent (convex) risk measure possesses Monotonicity and Translation Invariance.

For (iv), note that for a coherent (convex) risk measure \( \rho \), due to its dual representation, we have that \( E[p\cdot -X] \leq \rho(X) \leq \sup -X = -\inf X \) with extreme situations where \( \mathcal{P}_p \) equals a singleton \( \{\mathbb{P}\} \) or the whole \( \mathcal{P}_q \). Thus, if \( \rho + D \) is coherent (convex), hence limited, then \( D \) is lower range dominated because \( D(X) \leq -\rho(X) - \inf X \leq E[p\cdot X] - \inf X \). This concludes the proof. \( \square \)

Remark 3.2. As proved by Bäuerle and Müller (2006), in the presence of Law Invariance, Convexity and Monotonicity are equivalent to second order stochastic dominance for atom-less spaces. As Limitedness implies Monotonicity, in the presence of Convexity and Law Invariance, it also implies second order stochastic dominance.

Theorem 3.3. Let \( \rho : L^p \rightarrow \mathbb{R} \) be a coherent risk measure and \( D : L^p \rightarrow \mathbb{R}_+ \) a generalized deviation measure. Then:

(i) \( \rho + D \) is a coherent risk measure if and only if it fulfills Limitedness.

(ii) \( \rho \) and \( D \) are Fatou continuous and \( \rho + D \) limited if, and only if, \( \rho + D \) can be represented as \( \rho(X) + D(X) = \sup_{Q \in \mathcal{P}_{\rho+D}} E_Q[-X] \), where \( \mathcal{P}_{\rho+D} = \{Q \in \mathcal{P} : \frac{dQ}{dx} = \frac{dQ_\rho}{dx} + \frac{dQ_D}{dx} - 1, Q_\rho \in \mathcal{P}_\rho, Q_D \in \mathcal{P}_D\} \).

(iii) \( \rho \) and \( D \) are law invariant and \( \rho + D \) limited if, and only if, \( \rho + D \) can be represented as \( \rho(X) + D(X) = \sup_{m \in \mathcal{M}} \int_0^1 \rho^\alpha(X)m\alpha(\alpha) \), where \( \rho^\alpha(X) = \frac{1}{\alpha} \int_0^\alpha F_X^{-1}(u)du \) and \( \mathcal{M} = \{m \in \mathcal{P}_{[0,1]} : \int_{[0,1]} \frac{1}{\alpha} dm(\alpha) = \frac{dQ}{dx}, Q \in \mathcal{P}_{\rho+D}\} \).

Proof. We begin with (i). According to Proposition 3.1, if \( \rho + D \) is a coherent risk measure then it fulfills Limitedness. For the converse part, the Translation Invariance, Sub-additivity, and Positive Homogeneity of \( \rho + D \)
is a consequence of the individual axioms fulfilled by $\rho$ and $D$ individually by definition. As there is Limitedness by assumption, $\rho + D$ respects Monotonicity due to Proposition 3.1. Hence, it is a coherent risk measure.

For (ii), $\rho + D$ being limited implies it is a coherent risk measure, by the previous result. As $\rho$ and $D$ are Fatou continuous, by Theorem 2.5 they have representations with dual sets $\mathcal{P}_\rho$ and $\mathcal{P}_D$. Thus, $\rho + D$ is also Fatou continuous and has dual representation. We then obtain that

$$
\rho(X) + D(X) = \sup_{Q_\rho \in \mathcal{P}_\rho} E_{Q_\rho}[-X] + E_P[X] - \inf_{Q_D \in \mathcal{P}_D} E_{Q_D}[X]
$$

$$
= \sup_{Q_\rho \in \mathcal{P}_\rho, Q_D \in \mathcal{P}_D} \{ E_{Q_\rho}[-X] - E_P[-X] + E_{Q_D}[-X] \}
$$

$$
= \sup_{Q_\rho \in \mathcal{P}_\rho, Q_D \in \mathcal{P}_D} \left\{ E_P \left[ \frac{dQ_\rho}{dP} + \frac{dQ_D}{dP} - 1 \right] \right\}
$$

$$
= \sup_{Q \in \mathcal{P}_{\rho+D}} E_Q[-X],
$$

where $\mathcal{P}_{\rho+D} = \{ Q \in \mathcal{P} : \frac{dQ}{dP} = \frac{dQ_\rho}{dP} + \frac{dQ_D}{dP} - 1, Q_\rho \in \mathcal{P}_\rho, Q_D \in \mathcal{P}_D \}$. To show that $\mathcal{P}_{\rho+D}$ is composed by valid probability measures, we verify that for $Q \in \mathcal{P}_{\rho+D}$, $E_P \left[ \frac{dQ}{dP} \right] = E_P \left[ \frac{dQ_\rho}{dP} \right] + E_P \left[ \frac{dQ_D}{dP} \right] - E_P [1] = 1$. In addition, $\frac{dQ}{dP} \geq 0$ because of assuming the opposite would yield $E_P \left[ \frac{dQ}{dP} \right] < 0$, and therefore, $2 = E_P \left[ \frac{dQ}{dP} \right] + E_P \left[ \frac{dQ_\rho}{dP} \right] + E_P \left[ \frac{dQ_D}{dP} \right] - E_P [1] = 1$, a contradiction. Now, we assume that $\rho + D$ has such dual representation. Then $\rho + D$ is a Fatou continuous coherent risk measure that respects Limitedness. Reversing the deduction steps, one recovers the individual dual representations of both $\rho$ and $D$. By Theorem 2.5 these two measures possess Fatou continuity.

Regarding (iii), Kusuoka (2001) showed that coherent risk measures with Law Invariance and Fatou continuity axioms can have this kind of representation for some $\mathcal{M} \subset \mathcal{P}_{[0,1]}$. Results from Jouini et al. (2006) and Svindland (2010) guarantee that law-invariant convex risk measures defined in atomless spaces will automatically be Fatou continuous. Thus, $\rho + D$ can have this kind of representation because it is limited, then coherent. We can define a continuous variable $u \sim U(0,1)$ uniformly distributed between 0 and 1, such that $F_X^{-1}(u) = X$. For $Q \in \mathcal{P}_{\rho+D}$, we can obtain $\frac{dQ}{dP} = H(u) = \int_{(a,1]} \frac{1}{\alpha} \, dm(\alpha)$, where $H$ is a monotonically decreasing function and $m \in \mathcal{P}_{[0,1]}$. As $H$ is anti-monotonic in relation to $X$, one can reach the supremum in a dual rep-
presentation. Then we obtain that
\[
\rho(X) + \mathcal{D}(X) = \sup_{Q \in \mathcal{P}_{\rho + \mathcal{D}}} E_Q[-X]
\]
\[
= \sup_{Q \in \mathcal{P}_{\rho + \mathcal{D}}} E_P \left[ -X \frac{dQ}{dP} \right]
\]
\[
= \sup_{m \in \mathcal{M}} \left\{ \int_0^1 -F_X^{-1}(u) \left[ \int_{[u,1]} \frac{1}{\alpha} dm(\alpha) \right] du \right\}
\]
\[
= \sup_{m \in \mathcal{M}} \left\{ \int_{(0,1]} \left[ \frac{1}{\alpha} \int_0^\alpha -F_X^{-1}(u) du \right] dm(\alpha) \right\}
\]
\[
= \sup_{m \in \mathcal{M}} \left\{ \int_{(0,1]} \rho^\alpha dm(\alpha) \right\},
\]
where \( \mathcal{M} = \left\{ m \in \mathcal{P}_{(0,1]} : \int_{[u,1]} \frac{1}{\alpha} dm(\alpha) = \frac{dQ}{dP}, Q \in \mathcal{P}_{\rho + \mathcal{D}} \right\} \). We now assume that \( \rho + \mathcal{D} \) has such representation. Then it is a law-invariant coherent risk measure. This is only possible if both \( \rho \) and \( \mathcal{D} \) are law invariant. By (i), it is also limited. This concludes the proof of the theorem.

Assertions of Theorem 3.3 can be extended in the case where \( \rho \) is a convex risk measure and \( \mathcal{D} \) a convex deviation measure. For the law invariant case, Frittelli and Rosazza Gianin (2005) proved representations similar to those of Kusuoka (2001) for convex risk measures. The results of Theorem 3.3 can also be extended to the case where \( \rho \) and \( \mathcal{D} \) are co-monotone. In this scenario, \( \mathcal{M} \) becomes a singleton, as is the case of the spectral risk measures proposed by Acerbi (2002) and concave distortion functions, which are widely used in insurance. Grechuk et al. (2009) proved results linking these classes and axioms for generalized deviation measures. We state these two extensions without proof, because the deductions are quite similar to the coherent case.

**Theorem 3.4.** Let \( \rho : L^p \to \mathbb{R} \) be a convex risk measure and \( \mathcal{D} : L^p \to \mathbb{R}_+ \) a convex deviation measure. Then:

(i) \( \rho + \mathcal{D} \) is a convex risk measure if and only if it fulfills Limitedness.

(ii) \( \rho \) and \( \mathcal{D} \) are Fatou continuous and \( \rho + \mathcal{D} \) limited if, and only if, \( \rho + \mathcal{D} \) can be represented as \( \rho(X) + \mathcal{D}(X) = \sup_{Q \in \mathcal{P}_{\rho + \mathcal{D}}} \{ E_Q[-X] - \gamma_{\rho + \mathcal{D}}(Q) \} \), where \( \gamma_{\rho + \mathcal{D}} = \gamma_{\rho} + \gamma_{\mathcal{D}} \).

(iii) \( \rho \) and \( \mathcal{D} \) are law invariant and \( \rho + \mathcal{D} \) limited if, and only if, \( \rho + \mathcal{D} \) can be represented as \( \rho(X) + \mathcal{D}(X) = \sup_{m \in \mathcal{M}} \{ \int_0^1 \rho^\alpha(X) md(\alpha) - \gamma_{\rho + \mathcal{D}}(m) \} \).
Theorem 3.5. Let $\rho : L^p \rightarrow \mathbb{R}$ be a co-monotone coherent risk measure and $D : L^p \rightarrow \mathbb{R}_+$ a co-monotone generalized deviation measure. Then:

(i) $\rho + D$ is a co-monotone coherent risk measure if, and only if, it fulfills Limitedness.

(ii) $\rho$ and $D$ are Fatou continuous and $\rho + D$ limited if, and only if, $\rho + D$ can be represented as $\rho(X) + D(X) = \sup_{Q \in \mathcal{P}_{\rho+D}} E_Q[-X]$.

(iii) $\rho$ and $D$ are law invariant and $\rho + D$ limited if, and only if, $\rho + D$ can be represented as $\rho(X) + D(X) = \int_0^1 \rho^\alpha(X) md(\alpha)$, where $m \in \mathcal{P}_{[0,1]}$.

We then have results that formally guarantee that our class of risk measures indeed forms a sub-set of coherent risk measures. Thus, we have refined those measures that join both pillars of the risk concept, while possessing axiomatic properties from the most prominent class of risk measures. Any limited risk measure of the form $\rho + D$, with $\rho$ a coherent risk measure, and $D$ a generalized deviation measure, is contemplated by our results. Our results are also valid for the more flexible convex case, as well as for the more constrained co-monotone coherent case. The milestone is that in these cases we always obtain $D(X) \leq -\rho(X) - \inf X$, i.e., the dispersion term considers “financial information” from the interval between the loss represented by $\rho$ and the maximum loss $-\inf X = \sup -X$. Nonetheless, in the next section we explore a specific functional that presents a more directly applicable risk measure from our class.

4 Examples

Given a coherent risk measure $\rho$, Rockafellar et al. (2006) proved that it is possible to construct a lower range dominated generalized deviation measure as the conform $D(X) = \rho(X - E_\mathbb{P}[X]) = \rho(X) + E_\mathbb{P}[X]$. Further, we consider dispersion measured by the p-norm semi-deviation of results that represent losses greater than $\rho$ as the conform $D(X) = \|(X - \rho^*(X))\|_p, \rho^*(X) = -\rho(X)$. Because the objective is to penalize the risk measured by $\rho$, we only consider the dispersion of results that represent losses greater than this value. The role of the minus sign is simply an adjustment to place $\rho$ at the same level of $X$, because the former represents losses and the latter the results of an asset. In this way, given any coherent risk measure $\rho$ we introduce one new conform $\rho(X) + \beta\|(X - \rho^*(X))\|_p, 0 \leq \beta \leq 1$, which can be understood as a loss penalized by the dispersion of results worse than this conform.
role of $\beta$ is to choose the proportion of the dispersion that has to be included; thus, it functions similar to an aversion term. Hence, such a risk measure can be thought of as being more comprehensive because it yields more solid protection due to the penalty by dispersion. We now prove some theoretical properties for this functional form, based on the above-mentioned results.

**Theorem 4.1.** Let $\rho : L^p \to \mathbb{R}$ be a coherent risk measure. Then

(i) $\|(X - \rho^*(X))^-\|_p$ is a lower range dominated generalized deviation measure. It has dual set $P_D = \{Q \in P : \frac{dQ}{dP} \in L^q, E_P[\frac{dQ}{dP}] = 1, \frac{dQ}{dP} \geq 0, \sigma_P(\frac{dQ}{dP} - 1) \leq \frac{\|(X - \rho^*(X))^-\|_p}{\sigma_P(X)} \forall X \in L^p\}$ if $\rho$ is Fatou continuous.

(ii) $\rho(X) + \beta\|(X - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1$, is a coherent risk measure. It has dual set $P_{\rho+\beta D}, P_{\beta D} = \{Q \in P : \frac{dQ}{dP} = (1 - \beta) + \beta \frac{dQ_D}{dP}, Q_D \in P_D\}$ if $\rho$ is Fatou continuous.

(iii) If $\rho$ fulfills Law Invariance, then also do $\|(X - \rho^*(X))^-\|_p$ and $\rho(X) + \beta\|(X - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1$.

**Proof.** We begin by (i). First, it is necessary to prove the axioms of a lower range dominated generalized deviation measure. For Translation Insensitivity, we obtain for $X \in L^p, C \in \mathbb{R}$ that $\|(X + C) - \rho^*(X + C))^-\|_p = \|(X + (C - \rho^*(X))^-\|_p = \|(X - \rho^*(X))^-\|_p$. Non-negativity is obtained directly from the definition of the p-norm. For Sub-additivity, we use the well-known triangle inequality of both $(X)^-$ and the p-norm. Then, we have for $X, Y \in L^p$ that $\|(X + Y) - \rho^*(X + Y))^-\|_p \leq \|(X - \rho^*(X) + Y - \rho^*(Y))^-\|_p \leq \|(X - \rho^*(X))^- + Y - \rho^*(Y))^-\|_p \leq \|(X - \rho^*(X))^-\|_p + \|(Y - \rho^*(Y))^-\|_p$. In the case of Positive Homogeneity, for $X \in L^p$ and $\lambda \geq 0$ we obtain $\|\lambda(X - \rho^*(X))^-\|_p = \|\lambda(X - \rho^*(X))^-\|_p = \lambda\|(X - \rho^*(X))^-\|_p$. For Lower range Dominance, consider for $X \in L^p$ the sequence of inequalities $E_P[X] - \inf X \geq -\rho(X) - \inf X \geq (X - \rho^*(X))^-$. Keeping in mind that $\|C\|_p = C, \forall C \in \mathbb{R}_+$, performing this operation does not change the inequalities because all terms are non-negative. Thus, we obtain $\|(X - \rho^*(X))^-\|_p \leq -\rho(X) - \inf X \leq E_P[X] - \inf X$.

For the second step of (i), we use the proved axioms and previous results to obtain the dual representation. By assumption, $\rho$ and, hence, $\|(X - \rho^*(X))^-\|_p$ are Fatou continuous. Thus, Theorem 2.5 guarantees a dual representation, with measures $\{Q \in P : \frac{dQ}{dP} \in L^q, E_P[\frac{dQ}{dP}] = 1, \frac{dQ}{dP} \geq 0\}$. The probability measures of which the dual set is composed have to be restricted.
to those that

\[
\|(X - \rho^*(X))^-\|_p \geq E_P[X] - E_Q[X]
\]

\[
= E_P \left[ X \left( 1 - \frac{dQ}{dP} \right) \right]
\]

\[
= E_P[X] E_P \left[ 1 - \frac{dQ}{dP} \right]
\]

\[
+ \sigma_P(X) \sigma_P \left( 1 - \frac{dQ}{dP} \right) \text{corr}_P \left( X, 1 - \frac{dQ}{dP} \right)
\]

\[
= \sigma_P(X) \sigma_P \left( 1 - \frac{dQ}{dP} \right).
\]

Hence, \( \sigma_P \left( \frac{dQ}{dP} - 1 \right) \leq \frac{\|(X - \rho^*(X))^-\|_p}{\sigma_P(X)} \). Thus, we obtain the dual set \( P_D = \{ Q \in \mathcal{P} : \frac{dQ}{dP} \in L^p, E_P \left[ \frac{dQ}{dP} \right] = 1, \frac{dQ}{dP} \geq 0, \sigma_P \left( \frac{dQ}{dP} - 1 \right) \leq \frac{\|(X - \rho^*(X))^-\|_p}{\sigma_P(X)}, \forall X \in L^p \} \), as required.

Based on results from (i), the proof for (ii) becomes quite simple. Because \( \|(X - \rho^*(X))^-\|_p \) configures a lower range dominated generalized deviation measure, this also applies to \( \beta \|(X - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1 \). Thus, taking into account the deductions from (i), we obtain that \( \rho(X) + \beta \|(X - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1 \), is limited and, by Theorem 3.3, is a coherent risk measure. For the second part of (ii), the assumption is made that \( \rho \) and, hence, \( \rho(X) + \beta \|(X - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1 \), are Fatou continuous. Then, by Theorem 2.5 it possesses a representation with dual set \( \mathcal{P}_{\rho + \beta D} \). According to Rockafellar et al. (2006), generalized deviation measures such as \( \beta D, \beta \geq 0 \), have a dual set \( \mathcal{P}_{\beta D} = \{ Q \in \mathcal{P} : \frac{dQ}{dP} = (1 - \beta) + \beta \frac{dQ}{dP}, Q_D \in P_D \} \).

For (iii), if \( \rho \) is law invariant, then for \( X, Y \in L^p \) with \( F_X = F_Y \) we have that \( \rho(X) + \beta \|(X - \rho^*(X))^-\|_p = \rho(Y) + \beta \|(Y - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1 \). In this case, the representation from part (iii) of Theorem 3.3 applies. This concludes the proof of the Theorem.

\[ \square \]

Remark 4.2. The results of Theorem 4.1 can be extended to the case where \( \rho \) is a convex risk measure. In this case \( \|(X - \rho^*(X))^-\|_p \) is a lower range dominated convex deviation measure. Only the Convexity axiom is not present in the previous proof. Then, note that we have for \( X, Y \in L^p \) that \( \|(\lambda X + (1 - \lambda)Y) - \rho^*(\lambda X + (1 - \lambda)Y))^-\|_p \leq \|(\lambda(X - \rho^*(X)) + (1 - \lambda)(Y - \rho^*(Y)))^-\|_p \leq \|(\lambda(X - \rho^*(X)))^-\|_p + \|(1 - \lambda)(Y - \rho^*(Y)))^-\|_p = \lambda \|(X - \rho^*(X))^-\|_p + (1 - \lambda)\|(Y - \rho^*(Y))^-\|_p \). In this case, \( \rho(X) + \beta \|(X - \rho^*(X))^-\|_p, 0 \leq \beta \leq 1 \) is also a convex risk measure. For the co-monotone coherent case there is no extension because the p-norm does not fulfill this axiom.
In addition to an intuitive financial meaning, the functional form we propose possesses solid theoretical properties. Based on this structure, we argue that it is an important risk measurement tool for use in financial problems, such as practical risk management, capital requirement determination, optimal resource allocation, hedging strategies, and decision-making, as well as other areas of knowledge outside finance. We now present specific examples of \( \rho(X) + \beta \| (X - \rho^*(X))^- \|_p, 0 \leq \beta \leq 1 \) for choices of \( \rho \) that often appear in the literature or even in practical approaches. We consider the mean loss \( E_p[-X] \), the ES, and the Expectile-VaR (EVaR). For each case, we briefly describe the functional form, some properties, and comments.

**Example 4.3 (Mean loss).** The first example is obtained by choosing the mean loss \( \rho(X) = E_p[-X] \), which generates the one-sided risk measure \( \rho_p(X) = E_p[-X] + \beta \| (X - E_p[X])^- \|_p, 0 \leq \beta \leq 1 \). This kind of risk measure was studied in detail by Fischer (2003), who proved it is law invariant and coherent. By specifying \( p = 2 \) we are able to recover the widely known mean plus semi-standard deviation \( \rho_2(X) = E_p[-X] + \beta \| (X - E_p[X])^- \|_2 = E_p[-X] + \beta \sigma_p^-(X), 0 \leq \beta \leq 1 \), of which the theoretical properties were analyzed by Ogryczak and Ruszczyński (1999). The advantages of this risk measure are its simplicity and financial meaning. A limitation could be the fact it does not consider tail risks, because it focuses on a central tendency expectation.

**Example 4.4 (Expected Shortfall).** In this example we choose the ES, proposed by Acerbi and Tasche (2002), defined for continuous distributions as \( \rho(X) = ES^\alpha(X) = E_p[-X | X \leq F_X^{-1}(\alpha)], 0 \leq \alpha \leq 1 \). This risk measure represents the expected value of a loss, given that it is beyond the \( \alpha \)-quantile of interest. This quantile is directly linked to the VaR concept. ES is the most utilized coherent risk measure. It is also the foundation of the representations introduced in Kusuoka (2001) for law invariant coherent risk measures. Applying our functional form enables us to obtain, with a slight modification in the composition term \( \beta \), the risk measure proposed in Righi and Ceretta (2015), the Shortfall Deviation Risk (SDR), and conform \( SDR^\alpha(X) = ES^\alpha[X] + \beta \| (X - ES^\alpha(X))^- \|_p, 0 \leq \beta \leq 1 \). These authors studied the theoretical properties of the SDR in detail, apart from explaining its usefulness in distinct financial applications. In contrast to the previous example, SDR considers tail risks.

**Example 4.5 (Expectile Value at Risk).** Our next example regards a risk measure that has recently gained more attention, the EVaR. This measure is linked to the concept of an expectile, which is a generalized quantile function, given by \( \tau^\alpha = \arg \min_\theta E_p[(\alpha \cdot 1_{X \leq \theta})(X - \theta)^2], 0 \leq \alpha \leq 1 \), where \( 1_{\alpha} \) is an
indicator function with the value 1 if \( a \) is true and 0 otherwise. Bellini et al. (2014) proved that the EVaR, which is defined as \( EVaR^\alpha = -\tau^\alpha \), is a coherent risk measure for \( \alpha \leq 0.5 \). Ziegel (2014) showed that the EVaR is the only coherent risk measure, beyond the mean loss, that possesses the property of elicitability, which allows a function to have its forecasts evaluated. This concept can be very useful for practical risk management in the backtesting step. Moreover, Bellini and Di Bernardino (2015) presented empirical results that confirm the competitive performance of EVaR against other risk measures. Setting \( \rho(X) = EVaR^\alpha(X) \) in our functional form leads to the risk measure we refer to as Deviation EVaR (DEVaR)

\[
DEVaR^\alpha[X] = EVaR^\alpha[X] + \beta\|X - EVaR^\alpha(X)\|_p, 0 \leq \beta \leq 1.
\]

This risk measure has not appeared in the literature before, and it is a very interesting option for risk management, due to the very recent qualities discussed for expectiles.

5 Conclusion

In this paper we present a class of risk measures \( \rho + D \), where \( \rho \) is a coherent risk measure and \( D \) is a generalized deviation measure. Based on Limit-edness, an axiom we introduced, we proved that this set is a sub-class of coherent risk measures, by discussing some details such as dual representation. Our results can be widely extended to cases of convex or co-monotone coherent risk measures. Under this perspective, we present a specific formulation that generates from any coherent measure \( \rho \) a generalized deviation \( D \) based on dispersions of results worse than \( \rho \), which leads to a risk measure conform \( \rho(X) + \beta\|X - \rho^*(X)\|_p, 0 \leq \beta \leq 1 \). Finally, we present some examples of risk measures that lie in our proposed class. We argue that risk measures of this type can be very useful in financial applications such as risk measurement, capital requirement determination, optimal resource allocation, and hedging strategies, among others.

References


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