Dynamic selection and combination of conditional quantile forecasts, with application to value-at-risk modeling

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Abstract

We introduce an effective and computationally fast approach to combine conditional quantile forecasts. The approach uses the information of the relevant loss function for the quantile problem associated to each candidate model in order to define forecast combination weights in a dynamic fashion. Two important advantages of the proposed method are that i) does not require numerical optimization of the combination weights, which facilitates implementation when a large cross section of individual forecasts is considered and ii) the aggressiveness in the allocation across alternative forecasts and the trimming of worse forecasts can be easily calibrated with a single parameter. An empirical implementation of the method based on a large dimensional data set with 50 assets and on value-at-risk (VaR) forecasts obtained with a set of 16 alternative candidate models shows that the portfolio VaR forecasts based on the proposed method are accurate and outperform that of individual models in many instances.

Key words: backtesting, Basel II, conditional correlation models, market risk, loss function, volatility, risk management, volatility.

JEL classification: C22, C53, G17.

1 Introduction

At least since the 1995 amendment to the Basel Accord, value-at-risk (VaR) has been established as one of the most important risk measures designed to control and to manage market risk, and to determine the amount of capital subject to regulatory control (see Berkowitz & O’Brien, 2002; Santos et al., 2012). Not only because of the Basel accords, but also because of its popularity in the industry, VaR has attracted a considerable amount of attention. However, a number of deficiencies of VaR estimates computed by many financial institutions have been documented in the literature. Empirical evidence presented by Berkowitz & O’Brien (2002), Pérignon et al. (2008) and Pérignon & Smith (2010), for instance, suggest that financial institutions tend to overestimate their VaR during calm periods, thus leading to overly high capital requirement levels. In contrast, during stressed market conditions such as the 2007/2008 financial crisis, the size and the frequency of losses they have experienced were well above those predicted by their internal VaR models, suggesting underestimation of their VaR-based capital requirement levels (see...
Jorion, 2009). In order to address these issues and to avoid both under- and over-estimation of capital requirement levels, we propose to improve VaR measures by dynamically combining conditional quantile forecasts of several alternative VaR models using an effective and computationally fast approach which is very flexible and can be implemented in a variety of contexts.

A large number of studies have sought to find the most appropriate approach to model and to forecast the VaR of a portfolio of assets through backtesting a set of alternative specifications, and checking whether the number of VaR violations is adequate (see Berkowitz & O’Brien, 2002; Bauwens et al., 2006; McAleer & da Veiga, 2008b, among others). These studies, in general, focus on two main approaches to obtain the portfolio VaR: i) using a multivariate model for the system of individual asset returns; or ii) modeling portfolio returns directly using a univariate specification. Some authors conclude that it is probably better to adopt univariate models to estimate the VaR of a portfolio; see, for example, Berkowitz & O’Brien (2002), Bauwens et al. (2006), and McAleer (2009). On the other hand, McAleer & da Veiga (2008b) found mixed evidence about the comparative performance of univariate and multivariate models, and Santos et al. (2013) conclude that the multivariate approach provides more accurate VaR estimates.

Thus, in practical situations, risk managers face uncertainty about which VaR specification to use in each point in time. Moreover, conditional quantile measures such as the VaR are subject to changes, which amplify the uncertainty on which is the most appropriate approach to track the time-variation. These problems could be alleviated using forecast combinations, and yet very few references in the literature have analyzed the performance of this approach when applied to the problem of conditional quantile forecasts.

In fact, the literature on forecast combinations is extensive and points to the superiority of combined forecasts with respect to single models in many different contexts (see, for example, Timmermann, 2006, and references therein). Additionally, dynamic strategies for combining forecasts might also mitigate structural breaks and model misspecification (Pesaran & Timmermann, 2007), which could be useful to develop more robust risk management strategies. However, quantile forecast combination strategies have not been exploited very often. Although forecast combination schemes have been developed at least since Bates & Granger (1969), very few approaches are available for the problem of portfolio VaR. Giacomini & Komunjer (2005) pioneered this literature and considered the problem of combining two VaR forecasts in order to derive an encompassing test based on a method-of-moments (MM) estimation of the combination weights. Halbleib & Pohlmeier (2012) adopted a similar approach and devised a MM estimator for the combination weights. One limitation of the approach proposed in Giacomini & Komunjer (2005) and
Halbleib & Pohlmeier (2012) is that they are restricted to the case of only two candidate models. McAleer et al. (2013) considered an equally-weighted combination of the VaR forecasts of many candidate models in order to obtain a crisis-robust risk management strategy and found that the combined approach is able to outperform individual specifications.

The initial evidence provided in Giacomini & Komunjer (2005), Halbleib & Pohlmeier (2012), and McAleer et al. (2013) motivated us to further investigate if - and, more importantly, how - we can profit from combining alternative conditional quantile forecasts under a realistic setting given by a large number of alternative specifications to model the portfolio VaR. Our goal is to put forward an effective and computationally fast approach to combine VaR predictions. The approach uses the information of the relevant loss function for the quantile problem - the *tick* loss function studied in Giacomini & Komunjer (2005) - associated to each candidate model in order to define forecast combination weights in a dynamic fashion. The resulting combined VaR forecast also cope with statistical properties of dynamic quantile functions studies in Gouriéroux & Jasiak (2008).

Our approach for combining alternative VaR forecasts differs from the existing ones in at least three aspects. First, it is more general as it accommodates as many VaR forecasts as possible regardless the model or method used to obtain the forecasts, therefore overcoming the limitations of the existing VaR combination methods that focus only on two candidate forecasts. Second, the combination approach does not require numerical optimization of the combination weights, which greatly facilitates its practical implementation specially when a large cross section of individual forecasts is considered. Third, the approach nests the equally-weighted combination considered in McAleer et al. (2013) as a particular case. This specific model combination is found to outperform more sophisticated combination schemes in many contexts; see, for instance, Clemen (1989), De Menezes et al. (2000), Wallis (2011), and Genre et al. (2013). Timmermann (2006), for instance, argues that equal weights are optimal in situations with an arbitrary number of forecasts when the individual forecast errors have the same variance and identical pairwise correlations. Additionally, our approach also allows for a straightforward calibration of the influence of the best performing models in the combination. In particular, the aggressiveness in the allocation across alternative forecasts and the trimming of worse forecasts can be straightforwardly calibrated with a single parameter.

We test the effectiveness of the proposed forecast combination scheme using a data set composed of daily closing prices of the 50 mostly traded stocks belonging to the US stock market index SP100 over
a 10-year time span. The empirical applications works as follows. First, we implement a set of 16 most popular univariate and multivariate GARCH specifications, and obtain their corresponding day-ahead VaR forecasts for alternative confidence levels. The univariate specifications are the GARCH model of Bollerslev (1986), the asymmetric GJR-GARCH model of Glosten et al. (1993), the exponential GARCH (EGARCH) model of Nelson (1991), the threshold GARCH (TGARCH) model of Zakoian (1994), the asymmetric exponent GARCH (APARCH) model of Ding et al. (1993), asymmetric GARCH (AGARCH) model of Engle (1990), the non-linear asymmetric GARCH (NAGARCH) model of Engle & Ng (1993), and the fractionally integrated GARCH (FIGARCH) model of Baillie et al. (1996). The multivariate specifications are the exponentially weighted moving average, the rolling estimator of Foster & Nelson (1996), the scalar VECH of Bollerslev et al. (1988), the orthogonal GARCH of Alexander (2001), the constant conditional correlation of Bollerslev (1990), the dynamic conditional correlation model of Engle (2002) and its asymmetric version proposed in Cappiello et al. (2006), and the dynamic equicorrelation of Engle & Kelly (2012). Second, we obtain combined VaR forecasts according to our proposed approach and provide a backtesting analysis based on the unconditional, independence, and conditional coverage tests proposed in Christoffersen (1998). Finally, it is worth noting that even though our empirical illustration focus on parametric VaR models, our approach also accommodates simulation-based measures such as the Monte Carlo VaR; see Alexander (2009) for a review.

2 Portfolio VaR computation

In this Section, we describe alternative procedures to obtain portfolio VaR forecasts using single models as well as with the proposed forecast combination scheme. We also review the approach for testing the accuracy of VaR estimates. Throughout the paper, we focus on the portfolio VaR for a long position in which traders have bought the assets and wish to measure the risk associated to a decrease in their prices. Moreover, we consider an equally-weighted portfolio, which has been been extensively used in the empirical literature; see, for instance, Zaffaroni (2007), DeMiguel et al. (2009) and Santos et al. (2013).

2.1 Individual VaR forecasts

We denote $R_{t+h} = (r_{1,t+h}, \ldots, r_{N,t+h})'$ the vector of $h$-period returns (between $t$ and $t + h$) of the $N$ assets included in the portfolio. The portfolio return is given by $r_{p,t+h} = w_t' R_{t+h}$, where $w_t$ is the vector...
of portfolio weights to be determined at time $t$. The portfolio VaR at time $t$ for a given holding period $h$ and confidence level $\vartheta$ is given by the $\vartheta$-quantile of the conditional distribution of the bond portfolio return. Thus, $\text{VaR}_t(h, \vartheta) = F_{p,t+h}^{-1}(\vartheta)$, where $F_{p,t+h}^{-1}$ is the inverse of the cumulative distribution function of $r_{p,t+h}$. Throughout the paper we focus on the portfolio VaR for a holding period of $h = 1$ day at $\vartheta = \{1\%, 2.5\%, 5.0\%, 10\%\}$, which are the most common risk levels used to compute the VaR.

When computing the VaR, it is common to consider two alternative conditioning sets in order to obtain the cumulative distribution function of the portfolio returns. First, one can consider the distribution of portfolio returns conditional on past portfolio returns, i.e. the distribution of $r_{p,t}$ conditional on a linear combination of past values of portfolio returns, $w_{t-1}'R_t$. Alternatively, one can consider the distribution of $r_{p,t}$ conditional on the whole vector of past asset returns, $R_{t-1}$. The former case leads to a univariate model for the portfolio returns while the latter leads to a multivariate model; see Santos et al. (2013) for a discussion.

When the distribution of returns is expressed in terms of its two first conditional moments, the portfolio return can be represented as

$$r_{p,t} = \mu_{p,t} + \sigma_{p,t}z_{p,t}, \quad (1)$$

where the standardized unexpected returns $z_{p,t}$ are independent and identically distributed with mean equal to zero and unit variance and $\mu_{p,t}$ and $\sigma_{p,t}$ are the conditional mean and standard deviation of the portfolio returns, respectively. In this case, the portfolio conditional VaR is given by

$$\text{VaR}_{t,\vartheta} = \mu_{p,t} + \sigma_{p,t}q_{\vartheta}. \quad (2)$$

In the fully parametric specification, $q_{\vartheta}$ in (2) is given by the $\vartheta$-quantile of the assumed distribution of $z_{p,t}$. We assume that $\mu_{p,t}$ in (2) is constant over time and will be estimated with the sample mean of the portfolio returns. In practice, the dependence in the conditional means of portfolio returns, when present, is very weak. Consequently, assuming a constant mean is not going to affect the results on the VaR estimation.

The specification of $\sigma_{p,t}$ in (2) depends on whether we consider a univariate or a multivariate model. When computing the VaR using a univariate model, $\sigma_{p,t}$ is given by the variance of portfolio returns
conditional on past portfolio returns, i.e.

\[ \sigma^2_{p,t} = E \left[ (r_{p,t} - \mu_{p,t})^2 | r_{p,1}, ..., r_{p,t-1} \right]. \]  

On the other hand, when dealing with multivariate models, \( \sigma_{p,t} \) is given by the variance of the portfolio returns conditional on past returns, i.e.

\[ \sigma^2_{p,t} = E \left[ (r_{p,t} - \mu_{p,t})^2 | R_1, ..., R_{t-1} \right] = w_{t-1}' H_t w_{t-1}, \]  

where \( H_t = E[(R_t - \mu_t)(R_t - \mu_t)' | R_1, ..., R_{t-1}] \) is the positive definite conditional covariance matrix of \( R_t \), and \( \mu_t = E(R_t | R_1, ..., R_{t-1}) \).

It is important to notice that there exists a myriad of alternative methods to model and to forecast the portfolio VaR, apart from the parametric univariate and multivariate approaches described above. Popular approaches include the conditional autoregressive VaR (CAViaR) model proposed by Engle & Manganelli (2004), the historical simulation and filtered historical simulation (FHS) discussed in Barone-Adesi et al. (1998), Boudoukh et al. (1998), and Hull & White (1998), and the Monte Carlo VaR; see Alexander (2009) for a review. More recently, Francq & Zakoian (2015b) developed a theory on risk parameter estimation in volatility models with application to VaR, whereas Francq & Zakoian (2015a) considered the simultaneous estimation of conditional VaR at multiple risk levels. It is far from our objective to provide a review on the existing models and methods. In contrast, our objective is to provide a general approach to combine VaR forecasts using a relevant criteria for the quantile problem. Next Section details the proposed approach to combine VaR forecasts.

### 2.2 Combined VaR forecasts

In this section, we detail the proposed approach to combine VaR predictions in a dynamic fashion. As noted in the previous Section, the problem of computing the portfolio VaR often requires choosing a given univariate or multivariate specification for the volatility of portfolio returns. In most practical situations, however, conditional quantiles are time-varying and the investor faces uncertainty on which is the most appropriate approach to model and to forecast the portfolio VaR at a given point in time. For instance, should the investor consider a univariate or multivariate specification? Are heavy-tailed distributions more appropriate than the Gaussian distribution? How to handle this uncertainty? All
these questions are typically answered in the literature by horse-racing many single VaR models; see, for
instance, Berkowitz & O’Brien (2002), Bauwens et al. (2006), Kuester et al. (2006), McAleer & da Veiga
(2008b), McAleer (2009), and Santos et al. (2013). We, on the other hand, consider the possibility of
improving the the estimation of the portfolio VaR via combined forecasts. More specifically, we consider
the case in which there exists $M$ alternative VaR models. In this case, the combined VaR estimator,
$VaR_{t,\vartheta}^{\text{Comb}}$, is defined as

$$VaR_{t,\vartheta}^{\text{Comb}} = \lambda_1,t VaR_{t,\vartheta}^1 + \ldots + \lambda_M,t VaR_{t,\vartheta}^M,$$

(5)

where $VaR_{t,\vartheta}^m$ denotes the time $t$ portfolio VaR forecast (obtained at time $t-1$) for the $\vartheta$-quantile obtained
with the $m$-th candidate model and $\lambda_{m,t}$ is the corresponding combination weight. It is worth highlighting
to important aspects regarding the definition of the combined VaR estimate in (5). First, the proposed
combination is very flexible as it accommodates VaR estimates obtained with alternative approaches
including, but not limited to, parametric univariate and multivariate specifications as discussed in Section
2.1. The second important aspect is that we focus on the case in which $\sum_{m=1}^M \lambda_{m,t} = 1$ (i.e. convex
combinations), and $\lambda_{m,t} \geq 0 \forall m$ (non negative combination weights). This restriction is important since,
as shown in Gouriéroux & Jasiak (2008), a convex combination of quantile functions is also a quantile
functions, which means that the proposed VaR combination in (5) satisfies the monotonicity condition
with respect to the risk level defined in Gouriéroux & Jasiak (2008).

The most important aspect when combining alternative individual VaR forecasts is to specify a com-
bination vector $\lambda_t = \{\lambda_{1,t}, \ldots, \lambda_{M,t}\}$ in (5), that is, to decide how much weight to place in each VaR
forecast. In order to define the combination vector $\lambda_m$ in (5), let us initially recall the relevant loss
function for the quantile problem, which is often referred to as the tick loss function and is defined as

$$L_{t,\vartheta}^m = [\vartheta - I(e_t < 0)] e_t,$$

(6)

where $L_{t,\vartheta}^m$ is the loss associated to $VaR_{t,\vartheta}^m$, which is the time $t$ portfolio VaR estimate for the $\vartheta$-quantile
obtained with the $m$-th candidate model. $e_t = r_{p,t} - VaR_{t,\vartheta}^m$ and $I(\cdot)$ is an indicator function. Giacomini
& Komunjer (2005) show that the loss function in (6) is the implicit loss function whenever the object
of interest is a forecast of a particular $\vartheta$-quantile. In particular, Giacomini & Komunjer (2005) show
that requiring $\mathbb{E}_t[\vartheta - I(e_t < 0)] = 0$ is exactly equivalent to Christoffersen’s (1998) correct conditional
coverage criterion.¹

In the context of portfolio VaR modeling, the combined forecast in (5) should emphasize the individual VaR models that yield lower values for the loss function \( L_{t,\vartheta} \) in (6), which means more accurate VaR estimates, and penalize those that yield higher values for the loss function (that is, less accurate VaR estimates). Therefore, upon obtaining VaR estimates using each of the model candidates, the combination vector \( \lambda_{m,t} \) in (5) can be defined as

\[
\lambda_{m,t} = \frac{(1/L_{m,t-1,\vartheta})^\eta}{\sum_{m=1}^{M} (1/L_{m,t-1,\vartheta})^\eta}, \quad m = 1, \ldots, M.
\] (7)

Notice that the combination vector in (7) uses the information regarding the performance of each candidate model in terms of the tick loss function at time \( t-1 \) to obtain the combined VaR forecast for time \( t \) according to (5). The tuning parameter \( \eta \geq 0 \) determines how aggressively we adjust the combination weights in response to changes in the portfolio VaR obtained with each of the candidate models. As \( \eta \to 0 \) we recover the equally-weighted model combination, and as \( \eta \to \infty \) the weight on the model that yields the lowest loss function \( L_{m,t-1,\vartheta} \) approaches 1. Thus, large values of \( \eta \) can shrink the combination weights towards the best performing models. For example, Granger & Jeon (2004) argue that poor performing models can substantially worsen forecasting performance. The additional shrinkage implied via \( \eta \) could provide some forecast benefits via reducing the importance of the worst performing models.

For a sufficiently large value of \( \eta \), the definition in (7) boils down to a “recent best” forecast combination rule which consists in a dynamic selection of the model that delivered the best performance in the most recent time period.²

¹To see how the tick loss function in (6) works in practice, it is useful to consider a simple example involving two alternative VaR models. Suppose that the portfolio return in day \( t \) is -1% and that the VaR for long position in day \( t \) (forecasted in \( t-1 \)) for the \( \vartheta = 1\% \) level from the each of the two models are -2% and -6%, respectively. Obviously, for the first model there is a VaR violation since the portfolio return is lower than the predicted VaR whereas for the second there is not a VaR violation. As for the first model, the value of the tick loss function in (6) is \( L_1^t = (0.01 - (-2)) \approx 2 \) whereas for the second model the value is \( L_2^t = (0.01 - 0)^2 = 0.02 \). Therefore, according to the tick loss function, a model is more penalized when a VaR violation is observed. Moreover, the greater the discrepancy of the VaR with respect to the observed portfolio return, the greater the penalization.

²The combination vector in (7) can be undefined when \( L_{m,t-1,\vartheta} = 0 \) since in this case there is a division by zero. In practice, having \( L_{m,t-1,\vartheta} = 0 \) means that the \( m \)-th specification delivered a correctly specified VaR estimate with minimum loss according to the tick function in (6). Intuitively, this situation suggests that the \( m \)-th specification should be strongly emphasized in the combination vector. Whenever this is the case, the forecast combination vector in (7) is redefined as \( \lambda_{m,t} = 1/N_{L_{m,t-1,\vartheta}=0} \), where \( N_{L_{m,t-1,\vartheta}=0} \) is the number of candidate models that delivered correctly specified VaR estimates, and \( \lambda_{m,t} = 0 \) if \( L_{m,t-1,\vartheta} > 0 \). For instance, when \( N_{L_{t-1,\vartheta}=0} = 2 \), it means that there are two candidate models that delivered correctly specified VaR forecasts. Therefore, the resulting combined forecast consists in an equally-weighted combination of these two VaR forecasts. In our empirical implementation discussed in Section 3, observing \( L_{m,t-1,\vartheta} = 0 \) is very uncommon, which means that the combination vector defined in (7) is applicable to the vast majority of instances.
It is worth noting that even though the combination vector in (7) is defined in terms of a statistical loss function, the resulting combined forecast is also economically motivated since the tick loss function used to compute the combination vector is closely related to the conditional coverage criterion defined in Christoffersen (1998); see Giacomini & Komunjer (2005). This criterion is often employed by financial institutions to assess the adequacy of their VaR estimates for the purpose of computing the amount of capital subject to regulatory control.

2.3 Methodology for backtesting the VaR

An important issue related to VaR modeling is the backtesting, which is the analysis of past VaR violations, see Christoffersen (1998), Christoffersen et al. (2001), and Andersen et al. (2006). This analysis is often based on the hit sequence, which is a sequence of binary variables that denotes VaR violations and can be defined as

\[ I_t = \begin{cases} 
1 & \text{if } r_{p,t} < VaR_t \\
0 & \text{if } r_{p,t} \geq VaR_t 
\end{cases} \]

where \( r_{p,t} \) is the portfolio return at time \( t \). Clearly, the behavior of the hit sequence is of main interest. Risk managers are concerned with VaR violations and, equally importantly, with whether these violations are clustered in time or if they appear to be randomly sparse. Clustered violations indicate that the VaR model can be misspecified and can fail to predict the portfolio VaR in times of high volatility such as during financial crises. For risk measurement purposes, the accuracy of VaR estimates during financial turmoils is highly desirable. Christoffersen (1998) point out that the problem of determining the accuracy of the VaR can be reduced to the problem of determining whether the hit sequence satisfies two properties. The first is the unconditional coverage property, which states that the probability of realizing a loss in excess of the reported VaR must be precisely \( \vartheta \times 100\% \). To check if this is the case, one has to compute the empirical (or realized) hit rate, which is the number of times in which the observed portfolio returns is lower than the estimated VaR over the total number of periods analyzed, i.e. hit rate = \( \frac{1}{T} \sum_{t=1}^{T} I (r_{p,t} < VaR_t) \). For instance, when computing the VaR at the 1% nominal level, one would expect that in 1% of the cases the observed portfolio return should be lower than the estimated VaR (i.e. a hit rate of 1%). The second aspect is the independence property, which indicates whether two elements of the hit sequence are independent from each other. Intuitively, if previous VaR violations presage a future VaR violation then this points to a general inadequacy in the reported VaR measure. In order to test for these desired
properties, we adopt in this paper the approach proposed by Christoffersen (1998). The approach consists in using the hit sequence to test for independence, unconditional coverage, and conditional coverage (joint test for independence and unconditional coverage).

To test the independence in the hit sequence, a likelihood ratio test (LR) of the following form is conducted,

$$LR_{ind} = -2 \log \left[ \frac{L(\hat{\Pi}_2; I_1, I_2, ..., I_T)}{L(\hat{\Pi}_1; I_1, I_2, ..., I_T)} \right],$$  \hspace{1cm} (8)

where the numerator corresponds to the likelihood function of a first order Markov model estimated with the output sequence \{\hat{I}_t\} and the denominator is the likelihood function of a binary Markov chain, and \(\hat{\Pi}_1\) and \(\hat{\Pi}_2\) are the corresponding transition probability matrices given by

$$\hat{\Pi}_1 = \begin{bmatrix} \frac{n_{00}}{n_{00}+n_{01}} & \frac{n_{01}}{n_{00}+n_{01}} \\ \frac{n_{10}}{n_{10}+n_{11}} & \frac{n_{11}}{n_{10}+n_{11}} \end{bmatrix},$$  \hspace{1cm} (9)

and \(\hat{\Pi}_2 = (n_{01} + n_{11}) / (n_{00} + n_{10} + n_{01} + n_{11})\), where \(n_{ij}\) is the number of observations with value \(i\) followed by \(j\). The \(LR_{ind}\) test is asymptotically distributed as a \(\chi^2\) distribution with 1 degree of freedom.

To test the unconditional coverage, the null hypothesis is \(E[I] = \vartheta\) against the alternative \(E[I] \neq \vartheta\). The likelihood function under the null is \(L(p; I_1, I_2, ..., I_T) = (1 - \vartheta)^{n_0}(\vartheta)^{n_1}\) and under the alternative is \(L(\pi; I_1, I_2, ..., I_T) = (1-\pi)^{n_0}(\pi)^{n_1}\). The resulting LR test is \(LR_{uc} = -2 \log \left[ \frac{L(p; I_1, I_2, ..., I_T)}{L(\hat{\pi}; I_1, I_2, ..., I_T)} \right]\), where \(\hat{\pi} = n_1/(n_0 + n_1)\) and \(n_1\) and \(n_0\) are the number of occurrences of ones and zeros in the hit sequence, respectively. The \(LR_{uc}\) is distributed as a \(\chi^2\) distribution with 1 degree of freedom. Finally, to test the joint hypothesis of independence and unconditional coverage, a LR of the form is conducted,

$$LR_{cc} = 2 \log \left[ \frac{L(p; I_1, I_2, ..., I_T)}{L(\hat{\Pi}_1; I_1, I_2, ..., I_T)} \right],$$

which is referred to as the conditional coverage test. The \(LR_{cc}\) test is distributed as a \(\chi^2\) distribution with 2 degrees of freedom.

### 3 Empirical application

In this Section, we discuss the empirical application carried out in the paper. The empirical exercise aims at investigating the accuracy of VaR estimates obtained with the individual models discussed in
Section 3.2 as well as with the proposed forecast combination approach discussed in Section 2.2. For that purpose, we first present the data, the individual candidate models and the implementation details. Finally, we discuss the results.

3.1 Data

The data set consists of closing prices of the 50 mostly traded stocks belonging to the US stock market index SP100 from 01/01/2004 until 31/12/2013. The data set has $T = 2510$ observations. Returns are computed as the difference in log prices multiplied by 100. The tickers included in the sample (in alphabetical order) are: AAPL, ABT, AMGN, AXP, BAC, BMY, C, CAT, CMCSA, COP, CSCO, CVS, CVX, DIS, DOW, EBAY, EMC, F, FCX, GE, GILD, GS, HAL, HD, HPQ, INTC, JNJ, JPM, KO, LOW, MO, MRK, MS, MSFT, ORCL, PFE, PG, QCOM, SBUX, SLB, T, TXN, UNH, UNP, USB, VZ, WFC, WMT, XOM.

3.2 Individual candidate models

In order to implement the proposed combination approach in (5) and (7) it is necessary to implement a set of individual VaR models. For that purpose, we consider the problem of computing the VaR for a long position on an equally-weighted portfolio using parametric univariate and multivariate models as discussed in Section 2.1. We implement a set of 16 alternative GARCH-type models widely used in risk measurement applications such as Giot & Laurent (2004), Kuester et al. (2006), McAleer & da Veiga (2008a,b), McAleer et al. (2013), among many others. Next we discuss the specifications employed in our empirical exercise.

3.2.1 Univariate specifications

We implement eight alternative univariate specifications for the portfolio conditional standard deviation $\sigma_{p,t}$: the (symmetric) GARCH model of Bollerslev (1986), the asymmetric GJR-GARCH model of Glosten et al. (1993), the exponential GARCH (EGARCH) model of Nelson (1991), the threshold GARCH (TGARCH) model of Zakoian (1994), the asymmetric exponent GARCH (APARCH) model of Ding et al. (1993), asymmetric GARCH (AGARCH) model of Engle (1990), the non-linear asymmetric GARCH (NAGARCH) model of Engle & Ng (1993), and the fractionally integrated GARCH (FIGARCH) model of Baillie et al. (1996). The asymmetric specifications are useful to take into account the asymmetric
response of volatility to positive and negative positive returns. In all models we consider their simplest forms where the variance only depends on one lag of past returns and past conditional variances. This simple specification has shown to be the most relevant in empirical applications. Stationarity conditions of all these specifications are available in many existing papers and books; see, for instance, Hentschel (1995) and Francq & Zakoian (2011).

The parameters of the univariate GARCH models are estimated by quasi maximum likelihood (QML) assuming Gaussian likelihoods. Reviews of estimation issues, such as the choice of initial values, numerical algorithms, accuracy, as well as asymptotic properties are given by Berkes et al. (2003), Robinson & Zaffaroni (2006), Francq & Zakoian (2009), and Zivot (2009). It is important to note that even when the normality assumption is inappropriate, the QML estimator based on maximizing the Gaussian log-likelihood is consistent and asymptotically Normal provided that the conditional mean and variance functions of the GARCH model are correctly specified; see Bollerslev & Wooldridge (1992).

3.2.2 Multivariate specifications

We also implement eight multivariate GARCH specifications: the exponentially weighted moving average (EWMA) model, the optimal rolling estimator (ORE) of Foster & Nelson (1996) as implemented in Fleming et al. (2001, 2003), the scalar VECH specification of Bollerslev et al. (1988) along with the variance targeting technique as suggested in Engle & Mezrich (1996), the orthogonal GARCH (O-GARCH) model of Alexander (2001) with 3 principal components, the constant conditional correlation (CCC) model of Bollerslev (1990), the dynamic conditional correlation (DCC) model of Engle (2002), the asymmetric DCC (ASYDCC) of Cappiello et al. (2006), and the dynamic equicorrelation (DECO) model of Engle & Kelly (2012). A more detailed description of these specifications can be also found in Caldeira et al. (2015).

Multivariate GARCH models are typically estimated via quasi maximum likelihood (QML). However, this estimator is found to be severely biased in large dimensions; see, for instance, Engle et al. (2008) and Hafner & Reznikova (2012). In this paper, the parameters of the EWMA, ORE, and VECH specifications are estimated with the composite likelihood (CL) method proposed by Engle et al. (2008). As for the conditional correlation models, their estimation can be conveniently divided into volatility part and correlation part. The volatility part refers to estimating the univariate conditional variances which is done by QML assuming Gaussian innovations. The parameters of the correlation matrix in the DCC and
ASYDCC models are estimated using the CL method. As pointed out by Engle et al. (2008), the CL estimator provides more accurate parameter estimates in comparison to the two-step procedure proposed by Engle (2002), especially in large problems.

3.3 Implementation details

We obtain combined VaR forecasts according to (5) and (7) by considering alternative values for the calibration parameter $\eta$. In particular, we set $\eta = \{0, 1, 2, 5\}$. When $\eta = 0$ the combination corresponds to the equally-weighted case. When the value of $\eta$ increases, the combination shrinks towards the best performing models. Additionally, we also implement a “recent best” combination strategy discussed in Section 2.2 which consists in selecting the model that delivered the best performance in the most recent time period. This particular combination is obtained when $\eta \to \infty$.

Our focus is on the portfolio VaR for a holding period of $h = 1$ day at $\vartheta = \{1\%, 2.5\%, 5.0\%, 10\%\}$ confidence levels. In order to compute the portfolio VaR according to (2), and to be consistent with the estimation procedure of the GARCH-type models discussed in Section 3.2, we consider that the distribution of $z_{p,t}$ is the Gaussian. To obtain one-step-ahead forecasts of the individual models and to obtain the combined forecasts, we employ an expanding window procedure that works as follows. Departing from the first $t = 1500$ observations, all models described in section 3.2 are estimated and their corresponding one-step-ahead forecasts of the conditional covariance matrix are obtained.\footnote{We have performed robustness checks with respect to the choice of the initial estimation window. In particular, we also considered the cases in which $t = 1000$ and $t = 2000$. The results are similar to those reported here and are available upon request. We believe that choosing an initial estimation window of $t = 1500$ is appropriate considering that we are estimating multivariate GARCH specifications with a cross section dimension of 50 assets.} We compute one-step-ahead VaR predictions for each of the models according to the approach discussed in Section 2 and compute the combined VaR forecasts according to (5) and (7) for $\eta = \{0, 1, 2, 5\}$. Finally, we add one observation to the estimation window and repeat the process until the end of the data set is reached. We end up with a sample of $T - t = 1010$ out-of-sample VaR forecasts. The backtesting approach discussed in 2.3 is applied only to the out-of-sample observations.

3.4 Results
Table 1: Backtesting analysis: individual models and forecast combinations

The Table reports

<table>
<thead>
<tr>
<th></th>
<th>Panel A: 1% VaR</th>
<th>Panel B: 2.5% VaR</th>
<th>Panel B: 5% VaR</th>
<th>Panel C: 10% VaR</th>
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<td><strong>p-value</strong></td>
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3.5 Robustness checks
Table 2: Robustness checks: Backtesting analysis for a short portfolio position

The Table reports

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<td>DCC</td>
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References


